Quantum computation

Lectures 7-9

Measurement-based quantum computation
Circuit model

Initial state
100...0

Ancilla qubits
all in 10

One-qubit gates and
C-NOT (or C-SIGN)

Measurement in computational basis

Universal initial state |Ψ⟩

One- and two-qubit measurements
plus feedforward to subsequent measurements

Measurement-based QC:
Teleportation QC
Cluster-state QC
TQC
CSQC (1WQC)

Review

1. Measurement in computational basis

\( Z = M = M_z \)

actual result is \((-1)^a\)

\( P_a = |K_a|^2 \)

2. Measurement in another basis \( |\alpha⟩\)

\((UZU^+) |\alpha⟩ = UZ |\alpha⟩ = (-1)^a |\alpha⟩\)
Generalize to keeping measured qubits.

1. Measurement in computational basis

\[ a \left| \psi \right\rangle \rightarrow P_a \left| \psi \right\rangle \left| \psi \right\rangle \]

2. Measurement in another basis \( |1a\rangle \)

\[
\begin{align*}
|\psi\rangle &= \sum_{a} c_a |1a\rangle \otimes |\chi_a\rangle \\
&\rightarrow U_a |1a\rangle \otimes |\chi_a\rangle \\
&\rightarrow \sum_{a} c_a |1a\rangle \otimes |\chi_a\rangle
\end{align*}
\]

Relative state decomposition:

\[ \sum_{a} c_a |1a\rangle \otimes |\chi_a\rangle \rightarrow |a\rangle \otimes |\chi_a\rangle \]
Classical vs. quantum control

\[ |a\rangle \rightarrow U^a |a\rangle \rightarrow Z |a\rangle \rightarrow U |a\rangle \]

\[ |\psi\rangle = \sum_{a,b} c_{ab} |a\rangle |b\rangle \otimes |\psi_{ab}\rangle \rightarrow |a\rangle \otimes \sum_b c_{ab} |1\rangle |b\rangle \rightarrow |a\rangle \otimes \sum_b c_{ab} U^a |1\rangle |b\rangle \]

\[ \sum_{a,b} c_{ab} |a\rangle |b\rangle U^a |1\rangle |b\rangle \]

Putting 2 and 3 together

\[ U^Z U^+ \]

\[ U^Z U \]

\[ U^Z \]

\[ U^Z U^+ \]
Two-qubit measurements

\[ \overset{\text{H}}{\text{Z}} \quad \overset{\text{Z}}{\text{Z}} \quad \overset{10}{\text{a}} \quad \overset{10}{\text{b}} \]\n
\[ \overset{\text{U}}{\text{Z}} \quad \overset{\text{Z}_\text{I} \text{I}_\text{OZ}}{\text{Z}} \quad \overset{10}{\text{a}} \quad \overset{10}{\text{b}} \]

Measurement in computational basis

Some rule for measurements in another basis \( U|a,b> \)

\[ \overset{\text{U}}{\text{U}} \quad \overset{\text{Z}_\text{I} \text{I}_\text{OZ}}{\text{Z}} \quad \overset{10}{\text{a}} \quad \overset{10}{\text{b}} \]

Example:

\[ U = \Lambda(x) H \otimes I \]

Bell basis

\[ |\beta_{a,b}> = \frac{1}{\sqrt{2}} \sum_{a,b} (-1)^{a+b} |a_0 b_0, a_1 b_1> = \frac{1}{\sqrt{2}} (|00> + (-1)^a |11>) \]

\[ = |z^a \otimes z^b |\beta_{a,b}> = |z^a \otimes z^b |100> = |z^a \otimes z^b |100> \]

\[ \Lambda(x) H \otimes I \Lambda(x) = X \otimes X \]

\[ \Lambda(x) I \otimes Z \Lambda(x) = Z \otimes Z \]

Notation:

\[ \Lambda(x) \]

phase bit

\[ \Lambda(x) \]

parity bit

Notation: \( \Lambda(x) \quad \Lambda(x) \quad \Lambda(x) \)
We can also take a two-qubit measurement apart into two successive incomplete quantum measurements.

\[ P_a \psi \leftrightarrow Z \psi \]

\[ P_a = \sum_b |b \otimes a \otimes b_a \rangle \langle b \otimes a \otimes b_a | \]

measurement of parity

Same old rule to get to other bases:

\[ \psi \leftrightarrow U \otimes Z \otimes U^+ \]
Relation of incomplete 2-qubit measurement to 1-qubit measurement

Example: \( Z \otimes I = V(x) Z \otimes Z V^\dagger(x) \)

Pause to check:

\[
|\psi\rangle = \sum_{c,b} c_{ab} |c,b\rangle \quad \rightarrow \quad |a\rangle \otimes \sum_{b} c_{ab} |b\rangle
\]

\[
\sum_{c,b} c_{ab} |c \oplus b, b\rangle \quad \rightarrow \quad \sum_{b} c_{ab} |a \oplus b, b\rangle
\]

parity of this state is \( c \)

measurement of parity determines \( c = a \)
Tools for converting a circuit to measurements:

1. Replacing state preparation on a to-be-active wire with measurements.
2. Adding measurements on to-be-discarded qubits.
3. Moving unitaries through measurements by unitarily transforming the measurement basis.
4. Discarding terminal unitaries on a to-be-discarded qubit.
5. Changing quantum controls to classical controls through Z-measurements.

Example: Using output only on and wire, convert to measurements.

\[ U^+ = H \Lambda(Z) \]

\[ = \Lambda(Z) \times \Xi \Lambda(Z) = \Xi \Lambda(Z) \]

This is the only way to eliminate the Z-gates that involve to-be-active qubits.
Need another measurement to eat up the C-PHASE:

Can this be right? How can random result b in first circuit be of any use in the second?

Check on next page

Elaborating:

Note: The projections at measurements can be accompanied by arbitrary global phase changes. Not only are such phase changes physically meaningless, they are arbitrary and meaningless in the algebra of tracking states through measurements.
TQC

Single-qubit unitaries

Teleportation circuit

Tack these on.
Single-qubit unitary measurement circuit

$$|\psi\rangle$$

$$U^+XU \otimes X U^+Z U \otimes Z$$

$$X \otimes X \otimes Z \otimes Z$$

But we really don't want to do these corrections, so what if we don't. Then the output state of the bottom qubit is

$$X^b Z^c X^d Z^e U|\psi\rangle = (-1)^{ab+d} Z^a+c U|\psi\rangle$$

meaningless global phase change

4 possible Pauli errors. We don't correct them now, but rather keep track of them till the next gate involving this qubit, and remove them in the process of performing that gate. At the ultimate measurement, we take out the errors by measuring in a different Pauli basis.

If we do the classical controls, we have no errors; if we don't, we have an error that is known.
We end up with

\[ |\psi\rangle \]

But we can do better if we don't start with teleportation as the primitive.

Note that if \( U \) is a Pauli operator, we can simply incorporate it into the Pauli errors and not bother doing it. Moreover, if \( U \) conjugates Pauli errors to Pauli errors, i.e., is in the Clifford group (normalizer of the Pauli group), then we don't have to correct the errors; we can just map them forward to be corrected later.

A simpler approach: use cheap teleportation circuits
Need one more measurement to eat up the C-NOT.

Putting in a unitary, we have

Leaving out the corrections, our single-qubit-unitary measurement circuit up to Pauli errors is
Notice that

\[ 1\psi\rangle \overset{X}{\rightarrow} 1\alpha\rangle \quad \overset{Z^a}{\rightarrow} \quad 1\psi\rangle \]

\[ 10\rangle \overset{Z^a}{\rightarrow} 1\psi\rangle \overset{H}{\rightarrow} 1\alpha\rangle \quad 10\rangle \]

\[ 1\alpha\rangle \overset{Z^a}{\rightarrow} \quad 1\psi\rangle \]

\[ \overset{H}{\rightarrow} 1\alpha\rangle \quad 17\rangle \]

Two-qubit unitaries: \( C-\text{PHASE} \ \wedge(\mathbf{Z}) \)

\[ 10\rangle \overset{H}{\rightarrow} \quad 1\psi\rangle \]

Teleports middle 2 qubits to outer 2 qubits and applies C-PHASE.

\[ \wedge(\mathbf{Z}) X \otimes I \wedge(\mathbf{Z}) = X \otimes Z \]

\[ \wedge(\mathbf{Z}) I \otimes X \wedge(\mathbf{Z}) = Z \otimes X \]

\[ \wedge(\mathbf{Z}) Z \otimes I \wedge(\mathbf{Z}) = Z \otimes I \]

\[ \wedge(\mathbf{Z}) I \otimes Z \wedge(\mathbf{Z}) = I \otimes Z \]

Controlled operations become \( X^a Z^b \) on top and \( Z^a X^b \) on bottom (or opposite order for both).

This is preparation of the entangled state

\[ (I \otimes H) |\rho_{00}\rangle = (I \otimes H)(Z^a X^b) |\beta_{ab}\rangle = (Z^a Z^b) (I \otimes H) |\beta_{ab}\rangle \]

Do a measurement to prepare \((I \otimes H) |\beta_{ab}\rangle\) and then correct with \( Z^a \otimes Z^b \).
Input doesn't matter.

Doesn't matter.

Tack on appropriate measurement to set up the controlled gate.
Putting it all together and leaving out the Pauli corrections, we have

Actually the input state has errors $X^az^aX^aw$, and we need to correct these. The $Z$-errors change $c \rightarrow c + r$ and $d \rightarrow d + t$. The $X$-errors change $a \rightarrow a + s$ and $b \rightarrow b + w$. No change in measurement circuit.

Visualizing a measurement-based circuit

$\Lambda(Z)U_0H_1\psi$