3.1 $D=4$

(a) $F|j\rangle = \frac{1}{2}\sum_k e^{i\pi jk/2}|k\rangle$

$$\boxed{\begin{aligned} F|0\rangle &= \tfrac{1}{2}(|0\rangle+|1\rangle+|2\rangle+|3\rangle) \\ F|1\rangle &= \tfrac{1}{2}(|0\rangle+i|1\rangle-|2\rangle-i|3\rangle) \\ F|2\rangle &= \tfrac{1}{2}(|0\rangle-|1\rangle+|2\rangle-|3\rangle) \\ F|3\rangle &= \tfrac{1}{2}(|0\rangle-i|1\rangle-|2\rangle+i|3\rangle) \end{aligned}}$$

$P|0\rangle = |0\rangle$

$$P|1\rangle = \frac{1}{4}\Big(\cancel{|0\rangle}+|1\rangle+\cancel{|2\rangle}+|3\rangle+\cancel{i|0\rangle}-|1\rangle-\cancel{i|2\rangle}+|3\rangle$$
$$-\cancel{|0\rangle}+|1\rangle-\cancel{|2\rangle}+|3\rangle-\cancel{i|0\rangle}-|1\rangle+\cancel{i|2\rangle}+|3\rangle\Big)$$

$P|1\rangle = |3\rangle$

$P|2\rangle = |2\rangle$

$$P|3\rangle = \frac{1}{4}\Big(\cancel{|0\rangle}+|1\rangle+\cancel{|2\rangle}+|3\rangle-\cancel{i|0\rangle}+|1\rangle+\cancel{i|2\rangle}-|3\rangle$$
$$-\cancel{|0\rangle}+|1\rangle-\cancel{|2\rangle}+|3\rangle+\cancel{i|0\rangle}+|1\rangle-\cancel{i|2\rangle}-|3\rangle\Big)$$

$= |1\rangle$

$$\boxed{\begin{aligned} P|0\rangle &= |0\rangle \\ P|1\rangle &= |3\rangle \\ P|2\rangle &= |2\rangle \\ P|3\rangle &= |1\rangle \end{aligned}}$$

(b) Eigenstates of P:

$$\boxed{\begin{array}{ll} P|0\rangle = |0\rangle & +1 \\ P|2\rangle = |2\rangle & +1 \\ P\frac{1}{\sqrt{2}}(|1\rangle+|3\rangle) = \frac{1}{\sqrt{2}}(|1\rangle+|3\rangle) & +1 \\ P\frac{1}{\sqrt{2}}(|1\rangle-|3\rangle) = -\frac{1}{\sqrt{2}}(|1\rangle-|3\rangle) & -1 \end{array}}$$

(c) $F|\phi\rangle = e^{i\mu}|\phi\rangle$, $\mu = 0, \pi/2, \pi, 3\pi/2$

$$F^2|\phi\rangle = e^{2i\mu}|\phi\rangle$$

Any eigenvector of $|\phi\rangle$ with eigenvalue $\begin{pmatrix} \pm 1 \\ \pm i \end{pmatrix}$ is also an eigenvector of $P = F^2$ with eigenvalue $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The eigenvectors of F with eigenvalues $\begin{pmatrix} \pm 1 \\ \pm i \end{pmatrix}$ span the eigensubspace of P with eigenvalue $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\therefore \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$ is an eigenvector of F with eigenvalue $\pm i$.

$$\boxed{F\frac{1}{\sqrt{2}}(|1\rangle-|3\rangle) = i\frac{1}{\sqrt{2}}(|1\rangle-|3\rangle) \qquad i}$$

$$F\frac{1}{\sqrt{2}}(|1\rangle+|3\rangle)=\frac{1}{\sqrt{2}}(|0\rangle-|2\rangle)$$

$$F\frac{1}{\sqrt{2}}(|0\rangle-|2\rangle)=\frac{1}{\sqrt{2}}(|1\rangle+|3\rangle)$$

$$\Longrightarrow \boxed{\begin{aligned} &F\tfrac{1}{2}(|0\rangle-|2\rangle+|1\rangle+|3\rangle)=\tfrac{1}{2}(|0\rangle-|2\rangle+|1\rangle+|3\rangle) && +1\\ &F\tfrac{1}{2}(|0\rangle-|2\rangle-|1\rangle-|3\rangle)=-\tfrac{1}{2}(|0\rangle-|2\rangle-|1\rangle-|3\rangle) && -1\\ &F\tfrac{1}{\sqrt{2}}(|0\rangle+|2\rangle)=\tfrac{1}{\sqrt{2}}(|0\rangle+|2\rangle) && +1 \end{aligned}}$$

Qubit labeling:
$$|0\rangle=|00\rangle$$
$$|1\rangle=|01\rangle$$
$$|2\rangle=|10\rangle$$
$$|3\rangle=|11\rangle$$

Eigenvectors:

Product States

$$i \quad \frac{1}{\sqrt{2}}(|1\rangle-|3\rangle)=\frac{1}{\sqrt{2}}(|01\rangle-|11\rangle)=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|1\rangle=\boxed{|-\rangle|1\rangle}$$

$$+1 \quad \frac{1}{\sqrt{2}}(|0\rangle+|2\rangle)=\frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle=\boxed{|+\rangle|0\rangle}$$

$$+1 \quad \frac{1}{2}(|0\rangle-|2\rangle+|1\rangle+|3\rangle)=\frac{1}{2}(|00\rangle-|10\rangle+|01\rangle+|11\rangle)$$
$$=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|0\rangle+\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|1\rangle\right)$$
$$=\boxed{\frac{1}{\sqrt{2}}(|-\rangle|0\rangle+|+\rangle|1\rangle)}$$

$$-1 \quad \frac{1}{2}(|0\rangle-|2\rangle-|1\rangle-|3\rangle)=\boxed{\frac{1}{\sqrt{2}}(|-\rangle|0\rangle-|+\rangle|1\rangle)}$$

maximally entangled states

<u>But</u>, in the $+1$ eigensubspace, we can take arbitrary linear combinations. In a general pair of orthogonal eigenstates, both will be entangled, but neither will be maximally entangled.