Solution 4.3

(a) The parity-check matrix H has $n-1$ rows and n columns. It can be chosen to be the unit matrix for the first $n-1$ columns, with the last column being all 1s:

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ I_{n-1} & & \vdots \\ & & 1 \end{pmatrix} .$$

Two 1s per row make all rows orthogonal to G, i.e., $HG = 0$, and also make each row perform a two-bit parity check.

(b) The syndrome for no error is, as always, the zero vector, this time in $n-1$ dimensions. The syndromes for single-bit errors are, as always, the columns of H. The syndrome for a multi-bit error is, as always, a sum of the columns corresponding to the bits in error.

Sums of up to t of the first $n-1 = 2t$ columns give all $2t$-dimensional vectors with t or fewer 1s. For these errors, the 1s in the syndrome identify the bits in error.

Adding the last column to any $2t$-dimensional vector produces the negative image, i.e., takes 0s to 1s and 1s to 0s. Adding the last column to sums of up to $t-1$ of the first $n-1 = 2t$ columns gives all vectors with $t-1$ or fewer 0s, i.e., with $2t - (t-1) = t+1$ or more 1s. These all correspond to errors on t or fewer bits, and for these errors, the 0s in the syndrome identify the bits that are in error in addition to the error on the last bit.

We have now identified the syndromes for all the errors on t or fewer bits, and these syndromes take up all the vectors in the $(n-1)$-dimensional vector space of syndromes. No further errors can be corrected.