

Homework Problem 1.3

Discussion Friday, September 5

1.3 Probability simplex and statistical distance. Consider a random variable with D possible values, and let p_j denote the probability for value j . The space of probabilities, defined by

$$1 = \sum_{j=1}^D p_j ,$$

$$p_j \geq 0, \quad j = 1, \dots, D,$$

is a $(D - 1)$ -dimensional regular polyhedron in D -dimensional Cartesian space.

(a) Draw the probability simplex for $D = 2, 3$, and 4.

(b) Consider a fiducial probability distribution p_j and a nearby distribution $q_j = p_j + \delta p_j$. By expanding the relative entropy $H(\mathbf{q}||\mathbf{p})$ about the fiducial distribution \mathbf{p} , show that the first nonvanishing term in the relative entropy is

$$H(\mathbf{q}||\mathbf{p}) = \frac{2}{\ln 2} \sum_j \frac{(\delta p_j)^2}{4p_j} .$$

The sum on the right-hand side defines a Riemannian metric on the probability simplex, which is called the *Bhattacharya-Wootters statistical distance* or the *Fischer information*.

For neighboring probability distributions, statistical distance is a measure of how easily the two distributions can be distinguished by repeated trials. More generally, the relative entropy is a distance measure on the simplex for finitely separated distributions (but it is not a Riemannian distance measure because it is not symmetric in \mathbf{q} and \mathbf{p}); it measures how likely you are to mistake \mathbf{q} for \mathbf{p} in repeated trials. We will encounter these distance measures again in the quantum context.

(c) By introducing new coördinates $r_j = \sqrt{p_j}$, show that according to the statistical distance, the probability simplex is a portion of a sphere of unit radius.