2.4 Consider a two-level atom with Hamiltonian

$$H = \tfrac{1}{2}\hbar\Omega\mathbf{n}\cdot\boldsymbol{\sigma}\ .$$

The eigenvalues of this Hamiltonian are $\pm\frac{1}{2}\hbar\Omega$, and the eigenvectors are $|\mathbf{n}\rangle$ and $|-\mathbf{n}\rangle$. This is also the Hamiltonian of a spin-$\frac{1}{2}$ particle in a magnetic field that points in the direction $\mathbf{n}$. We know that the unitary dynamics of this Hamiltonian is a rotation on the Bloch sphere, but in this problem we want to look at the dynamics in terms of the probability amplitudes.

(a) *Derive* the following evolution equations for the probability amplitudes $c_0(t) = \langle 0|\psi(t)\rangle$ and $c_1(t) = \langle 1|\psi(t)\rangle$ for the atom to be in the upper and lower energy levels:

$$\begin{aligned}
\frac{dc_0}{dt} &= -\frac{i}{2}\Omega\big(n_3c_0 + (n_1 - in_2)c_1\big)\\
\frac{dc_1}{dt} &= -\frac{i}{2}\Omega\big(-n_3c_1 + (n_1 + in_2)c_0\big)\ .
\end{aligned}$$

(b) Use any method at your disposal to *solve* the equations in part (a) for $c_0(t)$ and $c_1(t)$ in terms of the initial amplitudes $c_0(0)$ and $c_1(0)$. (Hint: You can use any method you want, but it's probably easiest to use what you know about the unitary evolution operator for this Hamiltonian.)

(c) *Specialize* the answer of part (b) to the case $\mathbf{n} = \mathbf{e}_z$. This is the free evolution of a system whose energy eigenstates are $|\mathbf{e}_z\rangle = |0\rangle$ and $|-\mathbf{e}_z\rangle = |1\rangle$, i.e., the states at the top and the bottom of the Bloch sphere, with eigenvalues $\hbar\Omega/2$ and $-\hbar\Omega/2$. Generally, the free evolution of a two-state system is a rotation about the Bloch axis defined by the energy eigenstates.