2.5 **Two-level atom interacting with a classical field mode.** Consider a two-level atom with energy-level spacing $\hbar\Omega$ interacting with a single *classical* field mode with frequency $\omega = \Omega + \Delta$, where Δ is the detuning. The Hamiltonian for the two-level atom is

$$H(t) = \tfrac{1}{2}\hbar\Omega\sigma_3 + \hbar g_0\big(\sigma_- e^{i\omega t} + \sigma_+ e^{-i\omega t}\big) ,$$

where g_0 is a coupling constant that we can assume to be a positive real number. The second part of the Hamiltonian describes transitions from the upper atomic level to the lower (σ_-) and from the lower level to the upper (σ_+). Notice that this Hamiltonian is explicitly time dependent; the explicit time dependences $e^{\pm i\omega t}$ come from the harmonic time dependence of the classical field.

A general way of solving the Schrödinger equation,

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H(t)|\psi(t)\rangle ,$$

is to find the *unitary evolution operator* $U(t)$, which satisfies an operator Schrödinger equation,

$$i\hbar\frac{dU(t)}{dt} = H(t)U(t) ,$$

and in terms of which the time-dependent state vector is

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle .$$

(a) *Show* that the evolution operator for this Hamiltonian can be written as

$$U(t) = U_R U_{R'} ,$$

where $R = R_{\mathbf{e}_3}(\omega t)$ is a rotation by angle ωt about $\mathbf{e}_3$, and $R' = R_{\mathbf{n}}(rt)$ is a rotation by angle rt about the unit vector

$$\mathbf{n} = \frac{2g_0}{r}\mathbf{e}_1 - \frac{\Delta}{r}\mathbf{e}_3 ,$$

with

$$r = (4g_0^2 + \Delta^2)^{1/2} .$$

(b) Use the result of part (a) to *write* the evolution of the Bloch vector $\mathbf{S}(t) = \langle\boldsymbol{\sigma}(t)\rangle$ in terms of R and R'.

(c) Suppose now that the atom is initially in the upper state $|0\rangle$, i.e., $\mathbf{S}(0) = \mathbf{e}_3$. Use the result of part (b) to *derive* an explicit expression for $\mathbf{S}(t)$; describe the motion of $\mathbf{S}(t)$ on the Bloch sphere.

(d) Continuing to assume that the atom is initially in the upper state, *find* the amplitudes $c_0(t) = \langle 0|U(t)|0\rangle$ and $c_1(t) = \langle 1|U(t)|0\rangle$ to be in the upper and lower states at time t. *Derive* directly from these amplitudes the components of the Bloch vector $\mathbf{S}(t)$, and *show* that they agree with the results of part (c).

(e) *Specialize* your expression for $\mathbf{S}(t)$ to the case of no detuning ($\Delta = 0$), and for this case describe or draw the motion of $\mathbf{S}(t)$ on the Bloch sphere. (The frequency $2g_0$ is called the Rabi frequency.)