Phys 572 Quantum Information Theory **Fall 2014**

Homework Problem 4.1 **Discussion Friday, October 31**

4.1 **Neumark extension of a rank-one POVM.** Consider a POVM for a D-dimensional quantum system, which consists of $N \geq D$ rank-one POVM elements (i.e., operators proportional to one-dimensional projectors). One way to think of this POVM is as a measurement of one-dimensional orthogonal projectors (an ODOP) on an N-dimensional space. This problem develops this way of thinking, which is called the Neumark extension.

Let the (rank-one) POVM elements be denoted by $E_\alpha = |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|$, $\alpha = 1, \ldots, N$, where the vectors $|\overline{\psi}_\alpha\rangle$ are subnormalized, i.e., $0 < \mu_\alpha = \langle\overline{\psi}_\alpha|\overline{\psi}_\alpha\rangle \leq 1$. The corresponding normalized vectors are $|\psi_\alpha\rangle = |\overline{\psi}_\alpha\rangle/\sqrt{\mu_\alpha}$. The POVM satisfies a completeness relation

$$P = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| = \sum_{\alpha=1}^{N} \mu_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ ,$$

where P denotes the identity operator on the D-dimensional Hilbert space.

(a) *Show* that by adding $N - D$ dimensions to the Hilbert space, you can find an orthonormal set of vectors $|\hat{\psi}_\alpha\rangle$, $\alpha = 1, \ldots, N$, that project to the POVM elements in the original D dimensions, i.e., $P|\hat{\psi}_\alpha\rangle = |\overline{\psi}_\alpha\rangle$. (Hint: Expand the POVM elements in an arbitrary orthonormal basis, and show that the expansion coefficients form part of a unitary matrix.)

The original D-dimensional Hilbert space is a subspace of the extended Hilbert space, and P is the projector onto that subspace. The N-dimensional Hilbert space is called the *direct sum* of the original Hilbert space and the Hilbert space of the added dimensions, both of which are subspaces of the N-dimensional Hilbert space.

The orthonormal set $|\hat{\psi}_\alpha\rangle$ is a *Neumark extension* of the POVM. The support of a system state ρ is confined to the original D-dimensional space, i.e., $P\rho P = \rho$. For such states the POVM measurement statistics are the same as the statistics of a measurement of the Neumark extension, i.e.,

$$\begin{aligned}\langle\hat{\psi}_\alpha|\rho|\hat{\psi}_\alpha\rangle = \mathrm{tr}(\rho|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|) = \mathrm{tr}(P\rho P|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|) &= \mathrm{tr}(\rho P|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|P) \\ &= \mathrm{tr}(\rho|\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|) = \mathrm{tr}(\rho E_\alpha) = p_\alpha \ .\end{aligned}$$

We can thus regard a measurement of a rank-one POVM as an ODOP measurement on a higher-dimensional space.

It is worth noting that the vectors $(1 - P)|\hat{\psi}_\alpha\rangle = |\tilde{\psi}_\alpha\rangle$, which are subnormalized vectors in the added dimensions, can be used to form a POVM in the added dimensions, i.e.,

$$1 - P = \sum_\alpha |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| \ .$$

The orthonormal vectors $|\hat{\psi}_\alpha\rangle$ are also a Neumark extension of this POVM.

One of the most instructive examples of a Neumark extension is provided by the canonical POVM for measuring the phase of a harmonic oscillator. The POVM elements, $E_\phi = |\phi\rangle\langle\phi|/2\pi$, are specified by the oscillator phase ϕ, where the "phase states,"

$$|\phi\rangle \equiv \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \ ,$$

satisfy

$$\left\langle n\middle|\left(\int_0^{2\pi}\frac{d\phi}{2\pi}\,|\phi\rangle\langle\phi|\right)\middle|n'\right\rangle = \int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{i(n-n')\phi} = \delta_{nn'}$$

and thus resolve the identity P according to

$$P = \int_0^{2\pi}\frac{d\phi}{2\pi}\,|\phi\rangle\langle\phi| = \int_0^{2\pi} d\phi\, E_\phi \ .$$

The Neumark extension consists of adding to the Hilbert space an infinite set of states corresponding to negative values of n. The resulting Hilbert space is that of a two-dimensional rotor; the states $|n\rangle$ can thought of as eigenstates of angular momentum with eigenvalue $n\hbar$. In this enlarged Hilbert space, the extended phase states,

$$|\hat\phi\rangle \equiv \sum_{n=-\infty}^{\infty} e^{in\phi}|n\rangle \ ,$$

are the "angle eigenstates." They resolve the identity in the bigger space, i.e.,

$$I = \int_0^{2\pi}\frac{d\phi}{2\pi}\,|\hat\phi\rangle\langle\hat\phi| \ .$$

In contrast to the states $|\phi\rangle$, they are also δ-normalized:

$$\langle\hat\phi|\hat\phi'\rangle = \sum_{n=-\infty}^{\infty} e^{-in(\phi-\phi')} = 2\pi\delta(\phi-\phi') \ .$$

The resolution of the identity and the orthonormality relation are standard results from the theory of Fourier series on the interval $[0, 2\pi]$. The extended phase states are a Neumark extension because when projected down to the space of nonnegative n, they give the original phase states $|\phi\rangle$.

(b) Consider the three Bloch vectors,

$$\begin{aligned}
\mathbf{n}_1 &= \mathbf{e}_x \ , \\
\mathbf{n}_2 &= -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y \ , \\
\mathbf{n}_3 &= -\frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y \ ,
\end{aligned}$$

which satisfy $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = -1/2$ for any pair $\alpha \neq \beta$ and which point to the vertices of an equilateral triangle in the equatorial plane. The corresponding Hilbert-space vectors, $|\mathbf{n}_1\rangle$, $|\mathbf{n}_2\rangle$, and $|\mathbf{n}_3\rangle$, can be used to form a three-outcome, rank-one POVM called the *trine*. *Write* out the trine POVM elements, and *construct* a Neumark extension for it.

(c) The four Bloch vectors,

$$\begin{aligned}
\mathbf{n}_1 &= \mathbf{e}_3 \ , \\
\mathbf{n}_2 &= \sqrt{\frac{8}{9}}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_3 \ , \\
\mathbf{n}_3 &= -\sqrt{\frac{2}{9}}\mathbf{e}_1 + \sqrt{\frac{2}{3}}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3 \ , \\
\mathbf{n}_4 &= -\sqrt{\frac{2}{9}}\mathbf{e}_1 - \sqrt{\frac{2}{3}}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3 \ ,
\end{aligned}$$

satisfy $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = -1/3$ for any pair $\alpha \neq \beta$ and point to the vertices of a tetrahedron. The corresponding Hilbert-space vectors, $|\mathbf{n}_1\rangle$, $|\mathbf{n}_2\rangle$, $|\mathbf{n}_3\rangle$, and $|\mathbf{n}_4\rangle$, can be used to form a four-outcome POVM called, not surprisingly, the *tetrahedron*. *Write* out the tetrahedron POVM elements, and *construct* a Neumark extension for it.