Phys 572 Quantum Information Theory Fall 2014

Homework Problem 4.4 Discussion Friday, October 31

4.4 **Open-system dynamics. I. Master equations in Lindblad form.**

Suppose a system Q is in contact with an environment E and that as a consequence, its marginal density operator changes in the infinitesimal time step from t to $t+dt$ according to

$$\rho(t+dt) = \rho(t) + dt\,\mathcal{L}(\rho) = (\mathcal{I} + dt\,\mathcal{L})(\rho)\ , \tag{1}$$

where $\mathcal{I} = I \odot I$ is the identity superoperator and the superoperator generator $\mathcal{L}$ is assumed to be time independent. There is a strong assumption in Eq. (1), i.e., that the evolution of Q in each time step depends only on the density operator of Q at the beginning of that time step. This means that any correlations built up between the system and the environment in previous time steps are irrelevant to the present evolution of Q. Another way of saying this is that the system and environment are uncorrelated at the beginning of each time step and thus that $\mathcal{I} + dt\,\mathcal{L}$ is a trace-preserving quantum operation. This assumption is called the *Markov assumption*. The purpose of this problem is to find the general form of the superoperator generator $\mathcal{L}$ given the constraints of complete positivity and trace preservation.

We can write Eq. (1) as a differential equation, called the *master equation*,

$$\frac{d\rho}{dt} = \mathcal{L}(\rho)\ , \tag{2}$$

which can be integrated to give

$$\rho(t) = \mathcal{C}_t(\rho)\ , \qquad \mathcal{C}_t = e^{\mathcal{L}t}\ ,$$

where $\mathcal{C}_t$ is a time-dependent, trace-preserving quantum operation with initial condition $\mathcal{C}_{t=0} = \mathcal{I}$. As a quantum operation, $\mathcal{C}_t$ has a time-dependent Kraus decomposition,

$$\mathcal{C}_t = e^{\mathcal{L}t} = \sum_{\alpha=0}^{N-1} B_\alpha(t) \odot B_\alpha^\dagger(t)\ ,$$

where N is the total number of Kraus operators. The trace-preserving condition is that

$$I = \mathcal{C}_t^*(I) = \sum_{\alpha=0}^{N-1} B_\alpha^\dagger(t) B_\alpha(t)\ .$$

At $t = 0$, $\mathcal{C}_t$ becomes the identity superoperator:

$$I \odot I = \mathcal{I} = \mathcal{C}_{t=0} = \sum_{\alpha=0}^{N-1} B_\alpha(0) \odot B_\alpha^\dagger(0)\ .$$

Since this gives two different Kraus decompositions of the identity superoperator, the decomposition theorem for completely positive maps tells us that

$$B_\alpha(0) = V_{\alpha 0} I \ ,$$

where the complex numbers $V_{\alpha 0}$ are the zeroth column of a unitary matrix; i.e., they are normalized to unity,

$$1 = \sum_{\alpha=0}^{N-1} |V_{\alpha 0}|^2 \ .$$

Now consider an infinitesimal time interval dt, for which we have

$$\mathcal{C}_{dt} = \mathcal{I} + \mathcal{L}\, dt = \sum_{\alpha=0}^{N-1} B_\alpha(dt) \odot B_\alpha^\dagger(dt) \ . \tag{3}$$

We can separate the decomposition operators $B_\alpha(dt)$ into two classes.

1. The first class consists of those Kraus operators that go to a (nonzero) multiple of I as dt goes to zero, i.e., those for which $V_{\alpha 0} \neq 0$. There must be at least one such operator to produce the identity contribution to $\mathcal{C}_{dt}$, but there can be more than one. Suppose there are m of these operators; assign them the indices $\alpha = 0, \ldots, m-1$. To produce terms linear in dt in Eq. (3), the decomposition operators in the first class must have the form

$$B_\alpha(dt) = V_{\alpha 0} I + dt\, b_\alpha \ , \quad \alpha = 0, \ldots, m-1 \ .$$

2. The second class consists of those Kraus operators that go to zero as dt goes to zero, i.e., those for which $V_{\alpha 0} = 0$. These operators contribute only to the $dt\,\mathcal{L}$ part of $\mathcal{C}_{dt}$. Suppose there are n of these operators, and let them have the indices $\alpha = m, \ldots, m+n-1 = N-1$. To produce terms linear in dt in Eq. (3), the decomposition operators in this second class must have the form

$$B_\alpha(dt) = \sqrt{dt}\, b_\alpha \ , \quad \alpha = m, \ldots, N-1 \ .$$

The square root here is crucial.

You are now ready to bring the master equation (2) into a standard form.

(a) *Show* that the master equation (2) can be brought into the form

$$\frac{d\rho}{dt} = -i[h, \rho] + \frac{1}{2} \sum_{\alpha=1}^{n} \big(2 a_\alpha \rho a_\alpha^\dagger - a_\alpha^\dagger a_\alpha \rho - \rho a_\alpha^\dagger a_\alpha \big) \ . \tag{4}$$

Here h is a Hermitian operator that can be thought of as the system Hamiltonian. The operators a_α describe the effect of the environment. This is a *Lindblad form* of the master equation, and the operators a_α are called *Lindblad operators*. The result of your work up till now that *any* Markovian open-system evolution of a quantum system has this form.

(b) One often sees the master equation written in a more general form,

$$\frac{d\rho}{dt} = -i[h, \rho] + \frac{1}{2} \sum_{\alpha,\beta} A_{\alpha\beta} \left(2c_\alpha \rho c_\beta^\dagger - \rho c_\beta^\dagger c_\alpha - c_\beta^\dagger c_\alpha \rho \right) , \tag{5}$$

where there can be an arbitrarily large number of Lindblad operators c_α and $A_{\alpha\beta}$ is any (square) positive matrix. This more general version of the master equation is usually called the Lindblad form. *Show* that this general Lindblad form can be converted to the form (4).