**Phys 572 Quantum Information Theory Fall 2014**

**Homework Problem 7.1 Discussion Saturday, November 22**

7.1 **Cloning and the isotropic POVM.** The isotropic POVM for a $D$-dimensional Hilbert space has a POVM element, $dE_{|\phi\rangle} = \alpha|\phi\rangle\langle\phi|\,d\Gamma_{|\phi\rangle}$, for every ray $|\phi\rangle$ in projective Hilbert space, where $d\Gamma_{|\phi\rangle}$ is the unitarily invariant measure on projective Hilbert space and $\alpha$ is a positive constant, thus implying that all outcomes are equally weighted. The POVM satisfies a completeness relation

$$I = \int dE_{|\phi\rangle} = \alpha \int d\Gamma_{|\phi\rangle}\, |\phi\rangle\langle\phi| \; .$$

If the system is in state $\rho$, the probability to get outcome $|\phi\rangle$ in a measurement of the isotropic POVM is

$$dp(|\phi\rangle|\rho) = \mathrm{tr}(\rho\, dE_\phi) = \alpha\, d\Gamma_{|\phi\rangle}\, \langle\phi|\rho|\phi\rangle \; .$$

To do this problem, we need to be able to do at least one integral over the invariant measure. The unitarily invariant line element on projective Hilbert space, called the *Fubini-Studi metric*, measures lengths in terms of Hilbert-space angle: the distance $d\gamma$ between a normalized vector $|\phi\rangle$ and a nearby normalized vector $|\phi'\rangle = |\phi\rangle + |d\phi\rangle$ is given by $\cos d\gamma = |\langle\phi|\phi'\rangle| = |1+\langle\phi|d\phi\rangle|$. Hilbert-space angle is not changed by global phase changes, confirming that we are dealing with rays in projective Hilbert space. The Fubini-Studi line element is given by

$$d\gamma^2 = \sin^2 d\gamma = 1 - \cos^2 d\gamma = -2\mathrm{Re}(\langle\phi|d\phi\rangle) - |\langle\phi|d\phi\rangle|^2 \; .$$

Normalization of $|\phi'\rangle$ requires that

$$0 = \langle\phi'|\phi'\rangle - 1 = 2\mathrm{Re}(\langle\phi|d\phi\rangle) + \langle d\phi|d\phi\rangle \; ,$$

which gives

$$d\gamma^2 = \langle d\phi|d\phi\rangle - |\langle\phi|d\phi\rangle|^2 = \langle d\phi|d\phi\rangle - \big(\mathrm{Im}(\langle\phi|d\phi\rangle)\big)^2 = \langle d\phi_\perp|d\phi_\perp\rangle \; ,$$

where $|d\phi_\perp\rangle = |d\phi\rangle - |\phi\rangle\langle\phi|d\phi\rangle$ is the projection of the small displacement $|d\phi\rangle$ orthogonal to $|\phi\rangle$.

The normalized vectors in a $D$-dimensional Hilbert space make up the sphere of unit radius, $\mathcal{S}_{2D-1}$, in $2D$ real dimensions. The contribution $\langle d\phi|d\phi\rangle$ to the Fubini-Studi line element is the standard metric on this unit sphere. The quantity $\langle\phi|d\phi\rangle$ is the component of the small displacement along $|\phi\rangle$: the real part describes changes in normalization, and the imaginary part describes changes in phase. The real part disappears from the line element because of the normalization constraint; the square of the imaginary part is subtracted away to remove the contribution of phase changes, because a global phase change does not change the Hilbert-space angle between two vectors.

Given a particular normalized vector $|\psi\rangle$, any other normalized vector can be written as

$$|\phi\rangle = e^{i\delta}\big(\cos\theta|\psi\rangle + \sin\theta|\chi\rangle\big) \ .$$

where $\delta$ is a global phase, $\theta$ is a "polar angle" in the range $0 \le \theta \le \pi/2$, and $|\chi\rangle$ is a normalized vector orthogonal to $|\psi\rangle$. We can get rid of the global phase freedom by choosing $\delta = 0$, thus working with rays in projective Hilbert-space. A small change in $|\phi\rangle$ takes the form

$$|d\phi\rangle = d\theta\big(-\sin\theta|\psi\rangle + \cos\theta|\chi\rangle\big) + \sin\theta|d\chi\rangle \ ,$$

which gives

$$\begin{aligned}\langle\phi|d\phi\rangle &= \sin^2\theta\langle\chi|d\chi\rangle \ ,\\ \langle d\phi|d\phi\rangle &= d\theta^2 + \sin^2\theta\langle d\chi|d\chi\rangle \ .\end{aligned}$$

The resulting line element is

$$\begin{aligned}d\gamma^2 &= d\theta^2 + \sin^2\theta\big(\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2\big)\\ &= d\theta^2 + \sin^2\theta\big(\langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2\big) \ ,\end{aligned}$$

where $d\chi_\perp\rangle = |d\chi\rangle - |\chi\rangle\langle\chi|d\chi\rangle$ is the projection of $|d\chi\rangle$ orthogonal to $|\chi\rangle$.

The line element $\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2 = \langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2$ is the standard metric on the unit sphere, $\mathcal{S}_{2D-3}$, in $2D-2$ real dimensions, except that along one real dimension, corresponding to phase changes of $|\chi\rangle$, lengths are scaled by a factor $\cos\theta$. The $\sin^2\theta$ in the line element means that lengths along all $2D-3$ real dimensions of $\mathcal{S}_{2D-3}$ are scaled by a factor of $\sin\theta$. Thus the integration measure that goes with the Fubini-Studi metric is

$$d\Gamma_{|\phi\rangle} = \sin^{2D-3}\theta\cos\theta\, d\theta\, dS_{2D-3} \ ,$$

where $dS_{2D-3}$ is the standard measure on the unit sphere $\mathcal{S}_{2D-3}$. This form of the integration measure is useful for doing integrals over functions of $|\langle\phi|\psi\rangle| = \cos\theta$.

In doing this problem, you should never have to calculate an explicit form for the area of a sphere. Instead, the area $S_{2D-3}$ of $\mathcal{S}_{2D-3}$ can be left as a normalization constant, whose value cancels out of the ultimate answer to part (c).

(a) Using the completeness relation, determine the value of the positive constant $\alpha$.

(b) *Find* the value of the integral

$$\int d\Gamma_{|\phi\rangle}|\langle\phi|\psi\rangle|^2 \ .$$

(c) One strategy for approximate cloning of an arbitrary state $|\psi\rangle$ is to measure the isotropic POVM and then make copies of the result $|\phi\rangle$. *Find* the average squared fidelity of the copies.