2.2. Nielsen and Chuang Exercise 2.39 plus Part (4). Trace inner product for operators.

Part (1).

$$\begin{aligned}(A,B) &\equiv \mathrm{tr}(A^\dagger B)\\ &= \sum_k \langle e_k|A^\dagger B|e_k\rangle \qquad \text{(definition of trace)}\\ &= \sum_{k,j} \langle e_k|A^\dagger|e_j\rangle\langle e_j|B|e_k\rangle \qquad \text{(insert resolution of 1)}\\ &= \sum_{k,j} \langle e_j|A|e_k\rangle^*\langle e_j|B|e_k\rangle \qquad \text{(definition of adjoint)}\\ &= \sum_{j,k} A^*_{jk}B_{jk}\end{aligned}$$

If we consider the matrix elements of an operator to be the components of a vector, the only difference being that the matrix elements are labeled by two indices (which is no difference at all), it is clear that this has the same form as the standard inner product on a complex vector space.

Part (2). Since the operators have d^2 matrix elements (vector components), it is clear that the vector space L_V has dimension d^2.

Part (3). The operators $\tau_{jk} = |e_j\rangle\langle e_k|$, $j,k = 1,\ldots,d$, are a basis for the space of operators, since we can write any operator as

$$A = \sum_{j,k} A_{jk}|e_j\rangle\langle e_k|\ .$$

It is easy to show that this basis is orthonormal:

$$(\tau_{jk},\tau_{lm}) = \mathrm{tr}(\tau^\dagger_{jk}\tau_{lm}) = \mathrm{tr}(|e_k\rangle\langle e_j|e_l\rangle\langle e_m|) = \langle e_m|e_k\rangle\langle e_j|e_l\rangle = \delta_{jl}\delta_{km}\ .$$

These basis operators are not, however, Hermitian, except when $j = k$. But since $\tau_{kj} = \tau^\dagger_{jk}$, we can form the Hermitian combinations $\tau_{jk} + \tau_{kj}$ and $-i(\tau_{jk} - \tau_{kj})$ when $j \neq k$. So we end up with the following set of d^2 (orthonormal) Hermitian basis operators:

$$\begin{aligned}&\tau_{jj}, \quad j = 1,\ldots,d,\\ &\frac{1}{\sqrt{2}}(\tau_{jk}+\tau_{kj}), \quad j \le k,\\ &\frac{-i}{\sqrt{2}}(\tau_{jk}-\tau_{kj}), \quad j \le k.\end{aligned}$$

Part (4). The first thing to note is that such a basis cannot be orthonormal for the following reason. Two one-dimensional projectors, $P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ and $P_\beta = |\psi_\beta\rangle\langle\psi_\beta|$, are orthogonal with respect to the Hilbert-Schmidt product if

$$0 = (P_\alpha, P_\beta) = \mathrm{tr}(P_\alpha P_\beta) = \mathrm{tr}\big(|\psi_\alpha\rangle\langle\psi_\alpha|\psi_\beta\rangle\langle\psi_\beta|\big) = |\langle\psi_\alpha|\psi_\beta|^2\ ,$$

i.e., if the state vectors $|\psi_\alpha\rangle$ and $\psi_\beta\rangle$ are orthogonal. There are at most d state vectors in an orthonormal set, so one cannot find d^2 orthogonal one-dimensional projectors.

But we can find d^2 linearly independent, but not orthogonal one-dimensional projectors, and here's how. Define state vectors

$$
\begin{aligned}
|\chi_{jk}\rangle &= \frac{1}{\sqrt{2}}(|e_j\rangle + |e_k\rangle) , \\
|\xi_{jk}\rangle &= \frac{1}{\sqrt{2}}(|e_j\rangle + i|e_k\rangle) ,
\end{aligned}
$$

for all pairs of indices, i.e., for $j < k$. The one-dimensional projectors formed from these state vectors,

$$
\begin{aligned}
P_{jk}^{+1} &= |\chi_{jk}\rangle\langle\chi_{jk}| = \frac{1}{2}(\tau_{jj} + \tau_{kk} + \tau_{jk} + \tau_{kj}) , \\
P_{jk}^{+i} &= |\xi_{jk}\rangle\langle\xi_{jk}| = \frac{1}{2}(\tau_{jj} + \tau_{kk} - i\tau_{jk} + i\tau_{kj}) ,
\end{aligned}
$$

go together with the projectors $P_j = \tau_{jj}$ to make up a set of d^2 one-dimensional projectors. They are an operator basis because the operator basis τ_{jk} can be written in terms of them:

$$
\begin{aligned}
\tau_{jj} &= P_j , \quad j = 1, \ldots, d, \\
\tau_{jk} &= P_{jk}^{+1} + iP_{jk}^{+i} - \frac{1}{2}(1+i)(P_j + P_k) , \quad j < k, \\
\tau_{kj} &= P_{jk}^{+1} - iP_{jk}^{+i} - \frac{1}{2}(1-i)(P_j + P_k) , \quad j < k.
\end{aligned}
$$

That there is an operator basis of pure states is a surprisingly important fact that we will need later on. It is not true in real vector spaces; the i in the definition of $|\xi_{jk}\rangle$ is important. You can see this because the i allows one to represent operators that are antisymmetric matrices in the standard basis; without the i, there is no way to get the antisymmetric matrices.