Solution 3.5.

(a) Let c denote the $(d_A \times d_B)$-matrix whose matrix elements are c_{jk}. Make c a square matrix by adding rows or columns of zeroes as appropriate. What this amounts to is adding fake dimensions to the smaller-dimensional system and assuming that $|\Psi\rangle$ is orthogonal to the fake dimensions.

The singular-value decomposition says that there exist unitary matrices V and W such that $c = VDW$, where D is a diagonal matrix whose diagonal entries, $\sqrt{\lambda_j}$, are the singular values of c, i.e., the eigenvalues of $\sqrt{c^\dagger c}$ or of $\sqrt{cc^\dagger}$ (since these two matrices have the same eigenvalues, the number of nonzero singular values is no larger than the smaller of the dimensions). Written out in terms of matrix elements, we have

$$c_{jk} = \sum_{l,m} V_{jl} D_{lm} W_{mk} = \sum_l \sqrt{\lambda_l} V_{jl} W_{lk} \ . \tag{1}$$

Thus the state takes the form

$$|\Psi\rangle = \sum_l \sqrt{\lambda_l} \biggl(\sum_j V_{jl} |e_j\rangle \biggr) \otimes \biggl(\sum_k W_{mk} |f_k\rangle \biggr) \ . \tag{2}$$

The vectors in big parentheses are the Schmidt vectors for A and B.

(b) The composite density operator is

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{j,k,l,m} c_{jk} c^*_{lm} |e_j, f_k\rangle\langle e_l, f_m| \ , \tag{3}$$

so

$$\rho_A = \mathrm{tr}_B(\rho) = \sum_{j,k,l,m} c_{jk} c^*_{lm} |e_j\rangle\langle e_l| \underbrace{\langle f_m|f_k\rangle}_{=\,\delta_{mk}} = \sum_{j,l} |e_j\rangle\langle e_l| \biggl(\sum_k c_{jk} c^*_{lk} \biggr) \tag{4}$$

The requirement that $\rho_A = I_A/d$ be maximally mixed says that

$$\rho_A = \frac{1}{d} \sum_j |e_j\rangle\langle e_j| \ , \tag{5}$$

which implies that

$$\sum_k c_{jk} c^*_{lk} = \frac{1}{d} \delta_{jl} \ , \tag{6}$$

i.e., that $\sqrt{d}c$ is a unitary matrix. The singular values of a unitary matrix are all equal to 1, so the Schmidt coefficients of a maximally entangled state are all equal to $1/\sqrt{d}$.