Solution 4.4.

(a) It is easy to work out that, given the Kraus operators in the two classes, the super-operator generator has the form

$$\mathcal{L} = \sum_{\alpha=0}^{m-1} \left(V_{\alpha 0}^* b_\alpha \odot I + I \odot V_{\alpha 0} b_\alpha^\dagger \right) + \sum_{\alpha=m}^{N-1} b_\alpha \odot b_\alpha^\dagger \ , \tag{6}$$

and, hence, the master equation (2) looks like

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \sum_{\alpha=0}^{m-1} \left(V_{\alpha 0}^* b_\alpha \rho + \rho V_{\alpha 0} b_\alpha^\dagger \right) + \sum_{\alpha=m}^{N-1} b_\alpha \rho b_\alpha^\dagger \ . \tag{7}$$

In terms of the operators

$$\begin{aligned} a_0 &= \sum_{\alpha=0}^{m-1} V_{\alpha 0}^* b_\alpha \ , \\ a_\alpha &= b_{\alpha+m-1} \ , \quad \alpha = 1, \ldots, n, \end{aligned} \tag{8}$$

the master equation becomes

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = a_0 \rho + \rho a_0^\dagger + \sum_{\alpha=1}^{n} a_\alpha \rho a_\alpha^\dagger \ . \tag{9}$$

The next step is to write a_0 in terms of Hermitian real and imaginary parts, $-g/2$ and $-h$, i.e.,

$$a_0 = -g/2 - ih \ , \tag{10}$$

so that

$$a_0 \rho + \rho a_0^\dagger = -\frac{1}{2}(g\rho + \rho g) - i[h, \rho] \ . \tag{11}$$

Plugging this into the master equation gives us

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[h, \rho] - \frac{1}{2}(g\rho + \rho g) + \sum_{\alpha=1}^{n} a_\alpha \rho a_\alpha^\dagger \ . \tag{12}$$

The last thing we need to do is to enforce the trace-preserving constraint. We could do this directly in terms of the Kraus operators $B_\alpha(dt)$, but it is simpler just to say

$$0 = \frac{d\,\mathrm{tr}(\rho)}{dt} = \mathcal{L}(\rho) = \mathrm{tr}\left(\rho \left(-g + \sum_{\alpha=1}^{n} a_\alpha^\dagger a_\alpha \right) \right) \ . \tag{13}$$

Since this must hold for all ρ, we get

$$g = \sum_{\alpha=1}^{n} a_\alpha^\dagger a_\alpha \ , \tag{14}$$

which allows us to put the master equation into the desired form:

$$\frac{d\rho}{dt} = -i[h,\rho] + \frac{1}{2}\sum_{\alpha=1}^{n}\bigl(2a_\alpha\rho a_\alpha^\dagger - a_\alpha^\dagger a_\alpha\rho - \rho a_\alpha^\dagger a_\alpha\bigr) \ . \tag{15}$$

(b) We start with the general Lindblad form (5),

$$\frac{d\rho}{dt} = -i[h,\rho] + \frac{1}{2}\sum_{\alpha,\beta} A_{\alpha\beta}\bigl(2c_\alpha\rho c_\beta^\dagger - \rho c_\beta^\dagger c_\alpha - c_\beta^\dagger c_\alpha\rho\bigr) \ ,$$

where $A_{\alpha\beta}$ is any positive matrix. We can write the matrix A as the $A = L^2$, where $L = \sqrt{A}$ is a positive matrix. In terms of indices this is

$$A_{\alpha\beta} = \sum_\gamma L_{\alpha\gamma}L_{\gamma\beta} = \sum_\gamma L_{\alpha\gamma}L^*_{\beta\gamma} \ . \tag{16}$$

Defining new Lindblad operators,

$$a_\gamma = \sum_\alpha L_{\alpha\gamma}c_\alpha \ , \tag{17}$$

puts the master equation into the form (4)

$$\frac{d\rho}{dt} = -i[h,\rho] + \frac{1}{2}\sum_\gamma\bigl(2a_\gamma\rho a_\gamma^\dagger - a_\gamma^\dagger a_\gamma\rho - \rho a_\gamma^\dagger a_\gamma\bigr) \ .$$

In this straightforward way of doing things, we end up with same number of Lindblad operators as we started with. If A has a null subspace, we can reduce the number of Lindblad operators to the rank of A by writing A as $A = LU^\dagger UL$ and choosing U to map from the standard basis to the eigenbasis of A.