C.17.3. Using Bloch-sphere notation, we have

$$P_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{n}_1)$$

$$P_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{n}_2)$$

$E_1 = \frac{\alpha}{2}(1-\vec{\sigma}\cdot\vec{n}_2)$ Falsifies $|\psi_2\rangle$ Confirms $|\psi_1\rangle$

$E_2 = \frac{\beta}{2}(1-\vec{\sigma}\cdot\vec{n}_1)$ Falsifies $|\psi_1\rangle$ Confirms $|\psi_2\rangle$

$0 \le \alpha, \beta \le 1$

$$g_1 P_1 + g_2 P_2 = \frac{1}{2}\left(1 + \vec{\sigma}\cdot(g_1\vec{n}_1 + g_2\vec{n}_2)\right)$$

$$E_1 + E_2 = \frac{\alpha+\beta}{2}1 - \vec{\sigma}\cdot\frac{\alpha\vec{n}_2+\beta\vec{n}_1}{2}$$

$$= \frac{\alpha+\beta}{2}1 - \frac{1}{2}|\alpha\vec{n}_2+\beta\vec{n}_1|\,\vec{\sigma}\cdot\vec{m}$$

$$\vec{m} \equiv \frac{\alpha\vec{n}_2+\beta\vec{n}_1}{|\alpha\vec{n}_2+\beta\vec{n}_1|}$$

$E_3 = 1 - E_1 - E_2 \longleftarrow$ inconclusive result

$$P_{nd} = g_1\,\mathrm{tr}(E_3P_1) + g_2\,\mathrm{tr}(E_3P_2)$$

$$= 1 - \underbrace{\mathrm{tr}\left((E_1+E_2)(g_1P_1+g_2P_2)\right)}_{g_1\,\mathrm{tr}(E_1P_1)\,+\,g_2\,\mathrm{tr}(E_2P_2)}$$

$$P_{nd} = 1 - g_1\,\mathrm{tr}(E_1P_1) - g_2\,\mathrm{tr}(E_2P_2)$$

The problem is to minimize P_{nd} subject to the constraint that $E_1+E_2 \le 1$.

$$\mathrm{tr}(E_1P_1) = \frac{\alpha}{2}(1 - \vec{n}_1\cdot\vec{n}_2)$$

$$\mathrm{tr}(E_2P_2) = \frac{\beta}{2}(1 - \vec{n}_1\cdot\vec{n}_2)$$

$$\frac{1}{2}A = \sin^2\theta = 1 - |\langle\psi_1|\psi_2\rangle|^2$$

$$q_1\,\mathrm{tr}(E_1P_1) + q_2\,\mathrm{tr}(E_2P_2) = \frac{1}{2}(\alpha q_1 + \beta q_2)(1 - \vec{n}_1\cdot\vec{n}_2)$$

$$E_1 + E_2 = \frac{1}{2}\left(\alpha + \beta - |\alpha\vec{n}_2 + \beta\vec{n}_1|\right)|\vec{m}\rangle\langle\vec{m}| + \frac{1}{2}\left(\alpha + \beta + |\alpha\vec{n}_2 + \beta\vec{n}_1|\right)|-\vec{m}\rangle\langle-\vec{m}|$$

Eigenvalues of $E_1 + E_2$

$$|\alpha - \beta| \leq |\alpha\vec{n}_2 + \beta\vec{n}_1| \leq \alpha + \beta$$

Problem is to maximize $\alpha q_1 + \beta q_2$ subject to the constraint that $\frac{1}{2}(\alpha + \beta + |\alpha\vec{n}_2 + \beta\vec{n}_1|) \leq 1$. The max will occur on the boundary so

$$\frac{1}{2}\Big(\alpha + \beta + \underbrace{|\alpha\vec{n}_2 + \beta\vec{n}_1|}_{=\left(\alpha^2 + \beta^2 + 2\alpha\beta\,\vec{n}_1\cdot\vec{n}_2\right)^{1/2}}\Big) = 1$$

$$2 - (\alpha + \beta) = (\alpha^2 + \beta^2 + 2\alpha\beta\,\vec{n}_1\cdot\vec{n}_2)^{1/2}$$

$$4 - 4(\alpha + \beta) + \cancel{\alpha^2} + \cancel{\beta^2} + 2\alpha\beta = \cancel{\alpha^2} + \cancel{\beta^2} + 2\alpha\beta\,\vec{n}_1\cdot\vec{n}_2$$

$$1 - (\alpha + \beta) + \frac{1}{2}\alpha\beta(1 - \vec{n}_1\cdot\vec{n}_2) = 0$$

Maximize using a Lagrange multiplier:

$$f(\alpha,\beta,\mu) = \alpha g_1 + \beta g_2 + \mu\Big(1 - (\alpha+\beta) + \tfrac{1}{2}\alpha\beta\underbrace{(1-\vec{n}_1\cdot\vec{n}_2)}_{A}\Big)$$

$$0 = \frac{\partial f}{\partial \mu} = 1 - (\alpha+\beta) + \tfrac{1}{2}\alpha\beta A \qquad \nu = 1/\mu$$

$$0 = \frac{\partial f}{\partial \alpha} = g_1 + \mu\left(-1 + \tfrac{1}{2}\beta A\right) \Rightarrow \beta = \frac{2(1-g_1/\mu)}{A} = \frac{2(1-\nu g_1)}{A}$$

$$0 = \frac{\partial f}{\partial \beta} = g_2 + \mu\left(-1 + \tfrac{1}{2}\alpha A\right) \Rightarrow \alpha = \frac{2(1-g_2/\mu)}{A} = \frac{2(1-\nu g_2)}{A}$$

$$\alpha+\beta = \frac{2}{A}\left(2-\nu(g_1+g_2)\right) = \frac{2}{A}(2-\nu)$$

$$\alpha\beta = \frac{4}{A^2}\left(1-\nu(g_1+g_2)+\nu^2 g_1 g_2\right)$$

$$\tfrac{1}{2}\alpha\beta A = \frac{2}{A}\left(1-\nu+\nu^2 g_1 g_2\right)$$

$$1 = (\alpha+\beta) - \tfrac{1}{2}\alpha\beta A$$

$$= \frac{2}{A}\left(2-\cancel{\nu}-1+\cancel{\nu}-\nu^2 g_1 g_2\right)$$

$$= \frac{2}{A}\left(1-\nu^2 g_1 g_2\right)$$

$$A = 1-\vec{n}_1\cdot\vec{n}_2$$
$$= 1 - 2|\langle\psi_1|\psi_2\rangle|^2 + 1$$
$$= 2\left(1-|\langle\psi_1|\psi_2\rangle|^2\right)$$
$$= 2\sin^2\theta$$

$$1-\frac{A}{2} = 1-\tfrac{1}{2}(1-\vec{n}_1\cdot\vec{n}_2)$$
$$= \tfrac{1}{2}(1+\vec{n}_1\cdot\vec{n}_2)$$
$$= |\langle\psi_1|\psi_2\rangle|^2$$
$$\equiv \cos^2\theta$$
$$0 \le \theta \le \pi/2$$

$$\nu^2 = \frac{1-A/2}{g_1 g_2} \Longrightarrow \nu = \pm\sqrt{\frac{1-A/2}{g_1 g_2}} = \pm\frac{\cos\theta}{\sqrt{g_1 g_2}}$$

$$0 \le A/2 = \sin^2\theta \le 1$$

$$\alpha_{\mp} = \frac{2}{A}\Big(1 \mp \sqrt{\frac{g_2}{g_1}}\underbrace{\sqrt{1-A/2}}_{\cos\theta}\Big), \qquad \beta_{\mp} = \frac{2}{A}\Big(1 \mp \sqrt{\frac{g_1}{g_2}}\underbrace{\sqrt{1-A/2}}_{\cos\theta}\Big)$$

Let's assume $q_1 \geq q_2$, and define $a \equiv \sqrt{q_1/q_2}$.
It's clear that $\beta_- \leq \alpha_- \leq \alpha_+ \leq \beta_+$. We have to choose the signs so that $0 \leq \beta \leq 1$.

$$\beta_\mp = \frac{1 \mp a\cos\theta}{\sin^2\theta}$$

It is clear that $\beta_+ > 1$ except at $\theta = \pi/2$, where it equals β_-. So we only need to consider β_-. $\beta_- \geq 0$ when $a\cos\theta \leq 1$. When $a\cos\theta > 1$, $\beta_- < 0$, which means that there is no extremum within the allowed region $0 \leq \alpha, \beta \leq 1$. Thus the <u>minimum</u> (not an extremum), occurs at the boundary of the allowed region, i.e., at $\beta = 0$ and $\alpha = 1$. So, for $\boxed{q_1 \geq q_2}$

$|\langle\psi_1|\psi_2\rangle| = \cos\theta > 1/a = \sqrt{q_2/q_1}$: $\alpha = 1$, $\beta = 0$

$$\Updownarrow$$

$$q_1|\langle\psi_1|\psi_2\rangle|^2 > q_2$$

$$\alpha q_1 + \beta q_2 = q_1$$

$$\begin{aligned}(P_{nd})_{min} &= 1 - (\alpha q_1 + \beta q_2)\sin^2\theta \\ &= 1 - q_1 \sin^2\theta \\ &= 1 - q_1(1 - |\langle\psi_1|\psi_2\rangle|^2) \\ &= q_2 + q_1|\langle\psi_1|\psi_2\rangle|^2\end{aligned}$$

$E_1 = \frac{1}{2}(1 - \vec{\sigma}\cdot\vec{n}_2) = |-\vec{n}_2\rangle\langle-\vec{n}_2|$

$E_2 = 0$

$E_3 = |\vec{n}_2\rangle\langle\vec{n}_2| = P_2 = |\psi_2\rangle\langle\psi_2|$

Never decide the state is $|\psi_2\rangle$.

$$|\langle\psi_1|\psi_2\rangle| = \cos\theta \le 1/a = \sqrt{\eta_2/\eta_1}:$$

$$\eta_1 |\langle\psi_1|\psi_2\rangle|^2 = \eta_2$$

$$\alpha = \alpha_- = \frac{1 - \sqrt{\eta_2/\eta_1}\cos\theta}{\sin^2\theta}$$

$$\beta = \beta_- = \frac{1 - \sqrt{\eta_1/\eta_2}\cos\theta}{\sin^2\theta}$$

$$\alpha\eta_1 + \beta\eta_2 = \frac{1 - 2\sqrt{\eta_1\eta_2}\cos\theta}{\sin^2\theta}$$

$$(P_{nd})_{min} = 1 - (\alpha\eta_1 + \beta\eta_2)\sin^2\theta$$

$$= 2\sqrt{\eta_1\eta_2}\cos\theta$$

$$= 2\sqrt{\eta_1\eta_2}\,|\langle\psi_1|\psi_2\rangle|$$