Homework Problem 1.3

Probability simplex and statistical distance. Consider a random variable with D possible values, and let p_j denote the probability for value j. The space of probabilities, defined by

$$1 = \sum_{j=1}^{D} p_j \ ,$$

$$p_j \geq 0, \quad j = 1, \ldots, D,$$

is a $(D-1)$-dimensional regular polyhedron in D-dimensional Cartesian space.

(a) *Draw* the probability simplex for $D = 2$, 3, and 4.

(b) Consider a fiducial probability distribution p_j and a nearby distribution $q_j = p_j + \delta p_j$. By expanding the relative entropy $H(\mathbf{q}||\mathbf{p})$ about the fiducial distribution $\mathbf{p}$, *show* that the first nonvanishing term in the relative entropy is

$$H(\mathbf{q}||\mathbf{p}) = \frac{2}{\ln 2} \sum_j \frac{(\delta p_j)^2}{4p_j} \ .$$

The sum on the right-hand side defines a Riemannian metric on the probability simplex, which is called the *Bhattacharya-Wootters statistical distance* or the *Fischer information.*

For neighboring probability distributions, statistical distance is a measure of how easily the two distributions can be distinguished by repeated trials. More generally, the relative entropy is a distance measure on the simplex for finitely separated distributions (but it is not a Riemannian distance measure because it is not symmetric in $\mathbf{q}$ and $\mathbf{p}$); it measures how likely you are to mistake $\mathbf{q}$ for $\mathbf{p}$ in repeated trials. We will encounter these distance measures again in the quantum context.

(c) By introducing new coördinates $r_j = \sqrt{p_j}$, *show* that according to the statistical distance, the probability simplex is a portion of a sphere of unit radius.