

Homework Problem 2.4

Consider a two-level atom with Hamiltonian

$$H = \frac{1}{2}\hbar\Omega\mathbf{n} \cdot \boldsymbol{\sigma} .$$

The eigenvalues of this Hamiltonian are $\pm\frac{1}{2}\hbar\Omega$, and the eigenvectors are $|\mathbf{n}\rangle$ and $|\!-\mathbf{n}\rangle$. This is also the Hamiltonian of a spin- $\frac{1}{2}$ particle in a magnetic field that points in the direction \mathbf{n} . We know that the unitary dynamics of this Hamiltonian is a rotation on the Bloch sphere, but in this problem we want to look at the dynamics in terms of the probability amplitudes.

(a) Derive the following evolution equations for the probability amplitudes $c_0(t) = \langle 0|\psi(t)\rangle$ and $c_1(t) = \langle 1|\psi(t)\rangle$ for the atom to be in the upper and lower energy levels:

$$\begin{aligned} \frac{dc_0}{dt} &= -\frac{i}{2}\Omega(n_3c_0 + (n_1 - in_2)c_1) \\ \frac{dc_1}{dt} &= -\frac{i}{2}\Omega(-n_3c_1 + (n_1 + in_2)c_0) . \end{aligned}$$

(b) Use any method at your disposal to solve the equations in part (a) for $c_0(t)$ and $c_1(t)$ in terms of the initial amplitudes $c_0(0)$ and $c_1(0)$. (Hint: You can use any method you want, but it's probably easiest to use what you know about the unitary evolution operator for this Hamiltonian.)

(c) Specialize the answer of part (b) to the case $\mathbf{n} = \mathbf{e}_z$. This is the free evolution of a system whose energy eigenstates are $|\mathbf{e}_z\rangle = |0\rangle$ and $|\!-\mathbf{e}_z\rangle = |1\rangle$, i.e., the states at the top and the bottom of the Bloch sphere, with eigenvalues $\hbar\Omega/2$ and $-\hbar\Omega/2$. Generally, the free evolution of a two-state system is a rotation about the Bloch axis defined by the energy eigenstates.