

## Homework Problem 2.5

**Two-level atom interacting with a classical field mode.** Consider a two-level atom with energy-level spacing  $\hbar\Omega$  interacting with a single *classical* field mode with frequency  $\omega = \Omega + \Delta$ , where  $\Delta$  is the detuning. The Hamiltonian for the two-level atom is

$$H(t) = \frac{1}{2}\hbar\Omega\sigma_3 + \hbar g_0(\sigma_- e^{i\omega t} + \sigma_+ e^{-i\omega t}) ,$$

where  $g_0$  is a coupling constant that we can assume to be a positive real number. The second part of the Hamiltonian describes transitions from the upper atomic level to the lower ( $\sigma_-$ ) and from the lower level to the upper ( $\sigma_+$ ). Notice that this Hamiltonian is explicitly time dependent; the explicit time dependences  $e^{\pm i\omega t}$  come from the harmonic time dependence of the classical field.

A general way of solving the Schrödinger equation,

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H(t)|\psi(t)\rangle ,$$

is to find the *unitary evolution operator*  $U(t)$ , which satisfies an operator Schrödinger equation,

$$i\hbar \frac{dU(t)}{dt} = H(t)U(t) ,$$

and in terms of which the time-dependent state vector is

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle .$$

(a) Show that the evolution operator for this Hamiltonian can be written as

$$U(t) = U_R U_{R'} ,$$

where  $R = R_{\mathbf{e}_3}(\omega t)$  is a rotation by angle  $\omega t$  about  $\mathbf{e}_3$ , and  $R' = R_{\mathbf{n}}(rt)$  is a rotation by angle  $rt$  about the unit vector

$$\mathbf{n} = \frac{2g_0}{r}\mathbf{e}_1 - \frac{\Delta}{r}\mathbf{e}_3 ,$$

with

$$r = (4g_0^2 + \Delta^2)^{1/2} .$$

(b) Use the result of part (a) to write the evolution of the Bloch vector  $\mathbf{S}(t) = \langle \boldsymbol{\sigma}(t) \rangle$  in terms of  $R$  and  $R'$ .

(c) Suppose now that the atom is initially in the upper state  $|0\rangle$ , i.e.,  $\mathbf{S}(0) = \mathbf{e}_3$ . Use the result of part (b) to *derive* an explicit expression for  $\mathbf{S}(t)$ ; describe the motion of  $\mathbf{S}(t)$  on the Bloch sphere.

(d) Continuing to assume that the atom is initially in the upper state, *find* the amplitudes  $c_0(t) = \langle 0|U(t)|0\rangle$  and  $c_1(t) = \langle 1|U(t)|0\rangle$  to be in the upper and lower states at time  $t$ . *Derive* directly from these amplitudes the components of the Bloch vector  $\mathbf{S}(t)$ , and *show* that they agree with the results of part (c).

(e) *Specialize* your expression for  $\mathbf{S}(t)$  to the case of no detuning ( $\Delta = 0$ ), and for this case describe or draw the motion of  $\mathbf{S}(t)$  on the Bloch sphere. (The frequency  $2g_0$  is called the Rabi frequency.)