**Phys 572 Quantum Information Theory Spring 2017**

**Homework Problem 3.2**

**Pure-state density operators.** A density operator $\rho$ is a positive operator that is normalized to unity, i.e.,

$$\rho \geq 0 \;, \quad \mathrm{tr}(\rho) = 1 \;.$$

A pure-state density operator is, in addition, a one-dimensional projector, so it squares to itself, i.e., $\rho^2 = \rho$. It is not hard to see, however, that one doesn't need this entire operator condition: $\rho$ is a pure state if and only if

$$\rho \geq 0 \;, \quad \mathrm{tr}(\rho) = 1 \;, \quad \mathrm{tr}(\rho^2) = 1 \;. \tag{1}$$

A density operator clearly satisfies these conditions, so we only need to prove sufficiency. to do so, note that $\rho \geq 0$ means that $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j$, where the eigenvalues $\lambda_j$ are nonnegative. The normalization condition, $1 = \mathrm{tr}(\rho) = \sum_j \lambda_j$ implies that $\lambda_j \leq 1$ for all $j$ and, hence, $\lambda_j^2 \leq \lambda_j$ for all $j$, with equality if and only if $\lambda_j = 0$ or 1. Then the trace conditions imply that

$$1 = \sum_j \lambda_j^2 \leq \sum_j \lambda_j = 1 \;.$$

The outer equalities require that the inequality be saturated, and it can only be satisfied if one eigenvalue is equal to 1 and all the others are zero, thus making $\rho$ a pure state.

One can eliminate the positivity condition by noting that if $\rho$ is Hermitian, then requiring $\rho = \rho^2 \geq 0$ guarantees that $\rho$ is positive. So we have an alternative way to characterize a pure state: $\rho$ is a pure state if and only if

$$\rho = \rho^\dagger \;, \quad \rho^2 = \rho \;, \quad \mathrm{tr}(\rho) = 1 \;. \tag{2}$$

Another way to look at these conditions is that the first two imply that $\rho$ is a projection operator (and, hence, a positive operator), and the last condition makes the projection operator one-dimensional since the trace of a projection operator is its rank.

No matter how you cut it, however, conditions (1) and (2) involve an operator condition, $\rho \geq 0$ or $\rho^2 = \rho$, which is really $d^2$ scalar conditions. Thus it is perhaps surprising that one can reduce the necessary and sufficient conditions for a pure state to just two scalar conditions.

*Prove* that $\rho$ is a pure state if and only if

$$\rho = \rho^\dagger \;, \quad \mathrm{tr}(\rho^2) = 1 \;, \quad \mathrm{tr}(\rho^3) = 1 \;.$$

The proof is a nearly a one-liner so if you're doing something more involved, you're probably off in the wrong direction.