Homework Problem 3.4

Maximal violation of the CHSH Bell inequality. Consider two qubits, P and Q. Let $A = \boldsymbol{\sigma}_P \cdot \mathbf{a}$, $B = \boldsymbol{\sigma}_Q \cdot \mathbf{b}$, $C = \boldsymbol{\sigma}_P \cdot \mathbf{c}$, and $D = \boldsymbol{\sigma}_Q \cdot \mathbf{d}$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are unit vectors in three dimensions. We omit the subscripts P and Q on the Pauli operators in the following because ordering in tensor products indicates which system the Pauli operators apply to, but you should feel free to re-introduce these labels whenever it clarifies things. Now let

$$\begin{aligned}\mathcal{B} &= A \otimes B + C \otimes B + C \otimes D - A \otimes D \\ &= \boldsymbol{\sigma} \cdot \mathbf{a} \otimes \boldsymbol{\sigma} \cdot (\mathbf{b} - \mathbf{d}) + \boldsymbol{\sigma} \cdot \mathbf{c} \otimes \boldsymbol{\sigma} \cdot (\mathbf{b} + \mathbf{d}) \\ &= |\mathbf{b} - \mathbf{d}| \boldsymbol{\sigma} \cdot \mathbf{a} \otimes \boldsymbol{\sigma} \cdot \mathbf{f} + |\mathbf{b} + \mathbf{d}| \boldsymbol{\sigma} \cdot \mathbf{c} \otimes \boldsymbol{\sigma} \cdot \mathbf{g}\end{aligned}$$

be the *Bell operator*. The quantity we called S in our discussion of the CHSH inequality is the expectation value of the Bell operator, i.e., $S = \langle \mathcal{B} \rangle$. In the final form of the Bell operator, we introduce unit vectors $\mathbf{f}$ and $\mathbf{g}$, which lie along the directions of $\mathbf{b} - \mathbf{d}$ and $\mathbf{b} + \mathbf{d}$.

(a) *Show* that $|S| = |\langle \mathcal{B} \rangle| \leq 2\sqrt{2}$. This result, called *T'sirelson's inequality*, determines the maximal violation of the CHSH Bell inequality.

(b) *Find* the conditions for equality in T'sirelson's inequality. (Warning: This part is hard, which is probably why it is not included in Nielsen and Chuang's Problem 2.3, which suggests a less efficient way of proving the T'sirelson bound.)