Homework Problem 3.7

Schmidt-like decomposition for three qubits. Consider three qubits, A, B, and C, each of which has a basis labeled by $|0\rangle$ and $|1\rangle$.

(a) *Show* that an *arbitrary* pure state $|\Psi\rangle$ of the three qubits can be transformed to the following Schmidt-like form using local unitary operators on A, B, and C:

$$\cos\theta|0\rangle\otimes\Big(\underbrace{\cos\chi|0\rangle\otimes|0\rangle+\sin\chi|1\rangle\otimes|1\rangle}_{=|\phi_0\rangle}\Big)$$
$$+\sin\theta|1\rangle\otimes\Big(\underbrace{\cos\xi\big(\sin\chi|0\rangle\otimes|0\rangle-\cos\chi|1\rangle\otimes|1\rangle\big)+e^{i\delta}\sin\xi\big(\cos\eta|0\rangle\otimes|1\rangle+\sin\eta|1\rangle\otimes|0\rangle\big)}_{=|\phi_1\rangle}\Big)\,.$$

The states $|\phi_0\rangle$ and $|\phi_1\rangle$ are orthonormal states of BC. Five parameters, θ, χ, ξ, η, and δ, are necessary to specify an arbitrary three-qubit pure state; *determine* the range of these five parameters. [Hint: Schmidt decompose $|\Psi\rangle$ with respect to the division A vs. BC. Then Schmidt decompose one of the resulting Schmidt states of BC with respect to the division B vs. C, writing the other BC Schmidt state in the resulting Schmidt bases of B and C. Use the freedom to rephase states to reduce the number of parameters, and use the freedom to apply local unitaries to get into the standard basis.]

The presence of four terms in $|\phi_1\rangle$, instead of just the first two terms or the last two terms, prevents this from being a genuine three-qubit Schmidt decomposition. This illustrates why there is generally no three-particle Schmidt decomposition.

(b) *Find* the marginal density operators of the three qubits.