Phys 572 Quantum Information Theory Spring 2017

Homework Problem 3.8

Operators on a real vector space. The linear operators on a D-dimensional vector space V over the complex numbers form a D^2-dimensional complex vector space L_V. The adjoint $A^\dagger$ is defined by

$$\langle\phi|A^\dagger|\psi\rangle = \langle\psi|A|\phi\rangle^* \quad \text{for all vectors } |\psi\rangle \text{ and } |\phi\rangle.$$

Relative to an orthonormal basis $|e_j\rangle$, $j = 1, \ldots, D$, the matrix elements of the adjoint are

$$(A^\dagger)_{jk} = \langle e_j|A^\dagger|e_k\rangle = \langle e_k|A|e_j\rangle^* = A^*_{kj}\ .$$

A natural inner product on L_V is given by

$$(A, B) = \mathrm{tr}(A^\dagger B) = \sum_{j,k} A^*_{jk} B_{jk}\ .$$

The Hermitian operators $A = A^\dagger$ form a D^2-dimensional real vector space H_V, whose complexification is L_V. In a previous homework problem, you constructed a basis consisting of D^2 pure states for L_V and H_V.

The linear operators on a D-dimensional vector space V over the real numbers form a D^2-dimensional real vector space L_V. The transpose A^T is defined by

$$\langle\phi|A^T|\psi\rangle = \langle\psi|A|\phi\rangle \quad \text{for all vectors } |\psi\rangle \text{ and } |\phi\rangle.$$

Relative to an orthonormal basis $|e_j\rangle$, $j = 1, \ldots, D$, the matrix elements of the transpose are

$$(A^T)_{jk} = \langle e_j|A^T|e_k\rangle = \langle e_k|A|e_j\rangle = A_{kj}\ .$$

A natural inner product on L_V is given by

$$(A, B) = \mathrm{tr}(A^T B) = \sum_{j,k} A_{jk} B_{jk}\ .$$

Notice that you can't define the transpose of operators on a complex vector space, because transposition is inconsistent with linearity:

$$\begin{aligned} a\langle\psi|A|\phi\rangle + b\langle\psi|A|\chi\rangle &= \langle\psi|A\big(a|\phi\rangle + b|\chi\rangle\big) \\ &= \big(\langle\phi|a^* + \langle\chi|b^*\big)A^T|\psi\rangle \\ &= a^*\langle\phi|A^T|\psi\rangle + b^*\langle\chi|A^T|\psi\rangle \\ &= a^*\langle\psi|A|\phi\rangle + b^*\langle\psi|A|\chi\rangle\ . \end{aligned}$$

The complex conjugation in the definition of the adjoint is there precisely to take care of this problem. What this means is that if you want to use the transpose of a linear operator in a complex vector space, you have to define it relative to a particular orthonormal basis. The same definition works in any other orthonormal basis that is a *real* linear combination of the vectors in the original basis.

This problem deals with linear operators L_V on a real vector space V.

(a) *Show* that the symmetric operators $A = A^T$ form a subspace S_V of L_V. What is the dimension of S_V? States and observables are symmetric operators. *Show* that the antisymmetric operators $A = -A^T$ form a subspace A_V of L_V. What is the dimension of A_V?

(b) *Show* that any operator O can be written as a sum of a symmetric operator and an antisymmetric operator. This result shows that L_V is the *direct sum* of S_V and A_V.

(c) *Show* that there is no basis of pure states that spans L_V. Given an orthonormal basis $|e_j\rangle$, $j = 1, \dots, D$, construct a basis of pure states for S_V. *Explain* what is different in a complex vector space that allows the pure states to span L_V.

(d) A system described by a real two-dimensional vector space V is called a *rebit*. *Find* a basis of orthogonal operators for S_V and a basis of orthogonal operators for A_V.

(e) Now consider two rebits. *Find* a basis of orthonormal operators for the symmetric subspace and a basis of orthonormal operators for the antisymmetric subspace. Is it possible to determine the state of two rebits from the statistics of local measurements?