Homework Problem 4.2

Convex set of POVM elements. The POVM elements for a D-dimensional quantum system consist of the positive operators E that satisfy $0 \le E \le 1$. This is equivalent to saying that the POVM elements are the Hermitian operators whose eigenvalues lie between 0 and 1, inclusive. The POVM elements form a convex set; i.e., if E and F are POVM elements, then for $0 \le \lambda \le 1$, so is $\lambda E + (1-\lambda)F$.

(a) The POVM elements for a qubit can be written in the Pauli representation as

$$E = r1 + s\mathbf{n}\cdot\boldsymbol{\sigma}\ ,$$

where r and s are real numbers. *Find* the allowed values of r and s, and *describe* the convex set of qubit POVM elements. By suppressing one of the "spatial" dimensions in the Pauli representation, *draw* a picture of the set of qubit POVM elements. From the picture, *identify* the extreme points of the convex set.

(b) For a D-dimensional system, *show* that the extreme points of the convex set of POVM elements are the projection operators of all ranks (the rank is the dimension of the support, i.e., the dimension of the subspace onto which the projection operator projects), including the zero operator (projector onto the zero-dimensional subspace) and the unit operator (projector onto the entire Hilbert space). (Hint: It's easy to show that a projector can't be written as a convex combination of other POVM elements. To show that any POVM element E can be written as a convex combination of projectors, start with the eigendecomposition of E, and cleverly rewrite it as a convex combination of projectors. You will need to use the zero operator in the convex combination.)