

Homework Problem 4.4

Open-system dynamics. I. Master equations in Lindblad form.

Suppose a system Q is in contact with an environment E and that as a consequence, its marginal density operator changes in the infinitesimal time step from t to $t + dt$ according to

$$\rho(t + dt) = \rho(t) + dt \mathcal{L}(\rho) = (\mathcal{I} + dt \mathcal{L})(\rho) , \quad (1)$$

where $\mathcal{I} = I \odot I$ is the identity superoperator and the superoperator generator \mathcal{L} is assumed to be time independent. There is a strong assumption in Eq. (1), i.e., that the evolution of Q in each time step depends only on the density operator of Q at the beginning of that time step. This means that any correlations built up between the system and the environment in previous time steps are irrelevant to the present evolution of Q . Another way of saying this is that the system and environment are uncorrelated at the beginning of each time step and thus that $\mathcal{I} + dt \mathcal{L}$ is a trace-preserving quantum operation. This assumption is called the *Markov assumption*. The purpose of this problem is to find the general form of the superoperator generator \mathcal{L} given the constraints of complete positivity and trace preservation.

We can write Eq. (1) as a differential equation, called the *master equation*,

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) , \quad (2)$$

which can be integrated to give

$$\rho(t) = \mathcal{C}_t(\rho) , \quad \mathcal{C}_t = e^{\mathcal{L}t} ,$$

where \mathcal{C}_t is a time-dependent, trace-preserving quantum operation with initial condition $\mathcal{C}_{t=0} = \mathcal{I}$. As a quantum operation, \mathcal{C}_t has a time-dependent Kraus decomposition,

$$\mathcal{C}_t = e^{\mathcal{L}t} = \sum_{\alpha=0}^{N-1} B_{\alpha}(t) \odot B_{\alpha}^{\dagger}(t) ,$$

where N is the total number of Kraus operators. The trace-preserving condition is that

$$I = \mathcal{C}_t^*(I) = \sum_{\alpha=0}^{N-1} B_{\alpha}^{\dagger}(t) B_{\alpha}(t) .$$

At $t = 0$, \mathcal{C}_t becomes the identity superoperator:

$$I \odot I = \mathcal{I} = \mathcal{C}_{t=0} = \sum_{\alpha=0}^{N-1} B_{\alpha}(0) \odot B_{\alpha}^{\dagger}(0) .$$

Since this gives two different Kraus decompositions of the identity superoperator, the decomposition theorem for completely positive maps tells us that

$$B_\alpha(0) = V_{\alpha 0} I ,$$

where the complex numbers $V_{\alpha 0}$ are the zeroth column of a unitary matrix; i.e., they are normalized to unity,

$$1 = \sum_{\alpha=0}^{N-1} |V_{\alpha 0}|^2 .$$

Now consider an infinitesimal time interval dt , for which we have

$$\mathcal{C}_{dt} = \mathcal{I} + \mathcal{L} dt = \sum_{\alpha=0}^{N-1} B_\alpha(dt) \odot B_\alpha^\dagger(dt) . \quad (3)$$

We can separate the decomposition operators $B_\alpha(dt)$ into two classes.

1. The first class consists of those Kraus operators that go to a (nonzero) multiple of I as dt goes to zero, i.e., those for which $V_{\alpha 0} \neq 0$. There must be at least one such operator to produce the identity contribution to \mathcal{C}_{dt} , but there can be more than one. Suppose there are m of these operators; assign them the indices $\alpha = 0, \dots, m-1$. To produce terms linear in dt in Eq. (3), the decomposition operators in the first class must have the form

$$B_\alpha(dt) = V_{\alpha 0} I + dt b_\alpha , \quad \alpha = 0, \dots, m-1 .$$

2. The second class consists of those Kraus operators that go to zero as dt goes to zero, i.e., those for which $V_{\alpha 0} = 0$. These operators contribute only to the $dt \mathcal{L}$ part of \mathcal{C}_{dt} . Suppose there are n of these operators, and let them have the indices $\alpha = m, \dots, m+n-1 = N-1$. To produce terms linear in dt in Eq. (3), the decomposition operators in this second class must have the form

$$B_\alpha(dt) = \sqrt{dt} b_\alpha , \quad \alpha = m, \dots, N-1 .$$

The square root here is crucial.

You are now ready to bring the master equation (2) into a standard form.

- (a) Show that the master equation (2) can be brought into the form

$$\frac{d\rho}{dt} = -i[h, \rho] + \frac{1}{2} \sum_{\alpha=1}^n (2a_\alpha \rho a_\alpha^\dagger - a_\alpha^\dagger a_\alpha \rho - \rho a_\alpha^\dagger a_\alpha) . \quad (4)$$

Here h is a Hermitian operator that can be thought of as the system Hamiltonian. The operators a_α describe the effect of the environment. This is a *Lindblad form* of the master equation, and the operators a_α are called *Lindblad operators*. The result of your work up till now that *any* Markovian open-system evolution of a quantum system has this form.

(b) One often sees the master equation written in a more general form,

$$\frac{d\rho}{dt} = -i[h, \rho] + \frac{1}{2} \sum_{\alpha, \beta} A_{\alpha\beta} (2c_\alpha \rho c_\beta^\dagger - \rho c_\beta^\dagger c_\alpha - c_\beta^\dagger c_\alpha \rho), \quad (5)$$

where there can be an arbitrarily large number of Lindblad operators c_α and $A_{\alpha\beta}$ is any (square) positive matrix. This more general version of the master equation is usually called the Lindblad form. *Show* that this general Lindblad form can be converted to the form (4).