Homework Problem 5.3

Converting a Neumark extension into a measurement model. In this problem we consider an arbitrary N-element, rank-one POVM for a qubit. We denote the POVM elements by $E_\alpha = |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| = \mu_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$, $\alpha = 1, \ldots, N$, where the vectors $|\psi_\alpha\rangle$ are normalized, and the vectors $|\overline{\psi}_\alpha\rangle = \sqrt{\mu_\alpha}|\psi_\alpha\rangle$ are subnormalized, with $\mu_\alpha = \langle\overline{\psi}_\alpha|\overline{\psi}_\alpha\rangle$. The Neumark-extended vectors, denoted by $|\hat{\psi}_\alpha\rangle$, live in a Hilbert space of dimension N, which is the *direct sum* of the two-dimensional qubit Hilbert space and a Hilbert space of $N-2$ additional dimensions. The Neumark-extended vectors satisfy $P|\hat{\psi}_\alpha\rangle = |\overline{\psi}_\alpha\rangle$, where $P = \sum_\alpha E_\alpha$ is the projector onto the qubit Hilbert space.

We're going to construct the Neumark extension in the 2^n-dimensional tensor-product space of n qubits; the $n-1$ added qubits will become ancillas in our measurement model. The first problem we run into is that N is generally not a power of 2. To handle this, we do the following. We must have $2^n \geq N$, so we choose n so that 2^n is the smallest power of 2 greater than or equal to N. We then add $2^n - N$ POVM elements equal to 0 to our POVM so that formally we are dealing with a rank-one POVM with 2^n outcomes. The last $2^n - N$ outcomes always have zero probability, so they never occur. Since the index α now takes on 2^n values, we replace it by the symbol $\mathbf{y} = y_1 \ldots y_n$, which stands for an n-bit binary string.

We also have to pay attention to the following problem. A Neumark extension is constructed by adding dimensions to the system Hilbert space; the extended Hilbert space is the direct sum of the original system Hilbert space and the vector space of the added dimensions. When we add $n-1$ qubits to the original qubit, we aren't adding Hilbert-space dimensions to the original qubit's Hilbert space; instead, we get a Hilbert space that is the *tensor product* of n two-dimensional Hilbert spaces. The original qubit's Hilbert space is not a subspace of the n-qubit tensor-product space; it makes no sense to talk about projecting onto the original qubit's Hilbert space (think about this). The problem we are confronting here is that the Neumark extension uses a direct sum, whereas a measurement model uses a tensor product.

To handle this problem, we consider the system Hilbert space to be the two-dimensional Hilbert space spanned by the vectors $|0\rangle \otimes |0\rangle^{\otimes(n-1)}$ and $|1\rangle \otimes |0\rangle^{\otimes(n-1)}$, where the superscript $\otimes(n-1)$ stands for an $(n-1)$-fold tensor product on the added qubits. This two-dimensional space is a subspace of the tensor-product space. The POVM elements take the form $E_\mathbf{y} = |\overline{\psi}_\mathbf{y}\rangle\langle\overline{\psi}_\mathbf{y}| \otimes P_0^{\otimes(n-1)} = \mu_\mathbf{y}|\psi_\mathbf{y}\rangle\langle\psi_\mathbf{y}| \otimes P_0^{\otimes(n-1)}$, where $P_0 = |0\rangle\langle 0|$ is the projector onto the $|0\rangle$ state. The projector onto the system Hilbert space is

$$P = \sum_\mathbf{y} E_\mathbf{y} = \sum_\mathbf{y} |\overline{\psi}_\mathbf{y}\rangle\langle\overline{\psi}_\mathbf{y}| \otimes P_0^{\otimes(n-1)} = I \otimes P_0^{\otimes(n-1)} \,.$$

The Neumark-extended vectors $|\hat{\psi}_\mathbf{y}\rangle$ live in the tensor-product space of the n qubits and satisfy $P|\hat{\psi}_\mathbf{y}\rangle = |\overline{\psi}_\mathbf{y}\rangle \otimes |0\rangle^{\otimes(n-1)} = \sqrt{\mu_\mathbf{y}}|\psi_\mathbf{y}\rangle \otimes |0\rangle^{\otimes(n-1)}$.

Now we're ready to get started. The actual problem is pretty easy, provided you have understood the setting developed in the introductory material above.

(a) By considering the way the Neumark extension is constructed from the standard basis $|\mathbf{x}\rangle = |x_1 \ldots x_n\rangle$, where $x_j = 0$ or 1, *draw* an n-qubit circuit that corresponds to a measurement of the POVM.

(b) The circuit of part (a) is unsatisfactory as a measurement model because it involves a direct measurement on the original qubit, instead of just measurements on the ancilla qubits. *Construct* a proper measurement model by adding one more ancilla qubit to your circuit. *Draw* the resulting $(n+1)$-qubit circuit, and *determine* the Kraus operators for the measurement model.

(c) By controlling on the outputs of the n measurements in part (b), *modify* the circuit so that the Kraus operators are $A_{\mathbf{y}} = \sqrt{E_{\mathbf{y}}} = \sqrt{\mu_{\mathbf{y}}}|\psi_{\mathbf{y}}\rangle\langle\psi_{\mathbf{y}}| = |\psi_{\mathbf{y}}\rangle\langle\overline{\psi}_{\mathbf{y}}|$. (Hint: You will need a complicated controlled operation.)