Homework Problem 6.1

Complete positivity of unital trace-preserving qubit quantum operations. An arbitrary trace-preserving qubit quantum operation can be written as

$$\mathcal{A}(\rho) = V\mathcal{B}(U\rho U^\dagger)V^\dagger \ ,$$

where

$$\mathcal{B}(\rho) = \frac{1}{2}\big(1 + \boldsymbol{\sigma}\cdot(T\mathbf{S} + \mathbf{d})\big) \ ,$$

and where

$$T = \begin{pmatrix} t_x & 0 & 0 \\ 0 & t_y & 0 \\ 0 & 0 & t_z \end{pmatrix}$$

is a diagonal matrix that contracts (and perhaps inverts or reflects, depending on the number of negative diagonal entries) the Bloch sphere and $\mathbf{d}$ is a displacement of the contracted sphere. The initial and final unitaries, U and V, represent initial and final rotations of the Bloch sphere. Since these unitary transformations are completely positive, the complete positivity of $\mathcal{A}$ is determined by the complete positivity of $\mathcal{B}$.

In this problem we deal with the special case where there is no displacement of the Bloch sphere, i.e., $\mathbf{d} = 0$. In this case the operation takes the maximally mixed state to itself, i.e., $\mathcal{B}(1) = 1$ (such a quantum operation is called *unital*). Positivity of $\mathcal{B}$ requires the output states to be in the Bloch sphere, which means that the diagonal entries of T must satisfy $|t_j| \le 1$. Thus positivity alone allows the diagonal entries to lie anywhere within a cube of side length 2 centered at the origin. The goal of this problem is to explore the further restrictions imposed by complete positivity.

(a) *Find* the restrictions imposed by complete positivity, and *describe* or *draw* the region of allowed values of t_x, t_y, and t_z.

(b) What kinds of maps of the Bloch sphere are allowed by complete positivity when one axis is left unchanged, say, $t_z = 1$?

(c) The unital qubit quantum operations form a convex set. What are the extreme points of this set?