

## Homework Problem 7.1

**Cloning and the isotropic POVM.** The isotropic POVM for a  $D$ -dimensional Hilbert space has a POVM element,  $dE_{|\phi\rangle} = \alpha|\phi\rangle\langle\phi|d\Gamma_{|\phi\rangle}$ , for every ray  $|\phi\rangle$  in projective Hilbert space, where  $d\Gamma_{|\phi\rangle}$  is the unitarily invariant measure on projective Hilbert space and  $\alpha$  is a positive constant, thus implying that all outcomes are equally weighted. The POVM satisfies a completeness relation

$$I = \int dE_{|\phi\rangle} = \alpha \int d\Gamma_{|\phi\rangle} |\phi\rangle\langle\phi| .$$

If the system is in state  $\rho$ , the probability to get outcome  $|\phi\rangle$  in a measurement of the isotropic POVM is

$$dp(|\phi\rangle|\rho) = \text{tr}(\rho dE_{|\phi\rangle}) = \alpha d\Gamma_{|\phi\rangle} \langle\phi|\rho|\phi\rangle .$$

To do this problem, we need to be able to do at least one integral over the invariant measure. The unitarily invariant line element on projective Hilbert space, called the *Fubini-Studi metric*, measures lengths in terms of Hilbert-space angle: the distance  $d\gamma$  between a normalized vector  $|\phi\rangle$  and a nearby normalized vector  $|\phi'\rangle = |\phi\rangle + |d\phi\rangle$  is given by  $\cos d\gamma = |\langle\phi|\phi'\rangle| = |1 + \langle\phi|d\phi\rangle|$ . Hilbert-space angle is not changed by global phase changes, confirming that we are dealing with rays in projective Hilbert space. The Fubini-Studi line element is given by

$$d\gamma^2 = \sin^2 d\gamma = 1 - \cos^2 d\gamma = -2\text{Re}(\langle\phi|d\phi\rangle) - |\langle\phi|d\phi\rangle|^2 .$$

Normalization of  $|\phi'\rangle$  requires that

$$0 = \langle\phi'|\phi'\rangle - 1 = 2\text{Re}(\langle\phi|d\phi\rangle) + \langle d\phi|d\phi\rangle ,$$

which gives

$$d\gamma^2 = \langle d\phi|d\phi\rangle - |\langle\phi|d\phi\rangle|^2 = \langle d\phi|d\phi\rangle - (\text{Im}(\langle\phi|d\phi\rangle))^2 = \langle d\phi_{\perp}|d\phi_{\perp}\rangle ,$$

where  $|d\phi_{\perp}\rangle = |d\phi\rangle - |\phi\rangle\langle\phi|d\phi\rangle$  is the projection of the small displacement  $|d\phi\rangle$  orthogonal to  $|\phi\rangle$ .

The normalized vectors in a  $D$ -dimensional Hilbert space make up the sphere of unit radius,  $\mathcal{S}_{2D-1}$ , in  $2D$  real dimensions. The contribution  $\langle d\phi|d\phi\rangle$  to the Fubini-Studi line element is the standard metric on this unit sphere. The quantity  $\langle\phi|d\phi\rangle$  is the component of the small displacement along  $|\phi\rangle$ : the real part describes changes in normalization, and the imaginary part describes changes in phase. The real part disappears from the line element because of the normalization constraint; the square of the imaginary part is subtracted away to remove the contribution of phase changes, because a global phase change does not change the Hilbert-space angle between two vectors.

Given a particular normalized vector  $|\psi\rangle$ , any other normalized vector can be written as

$$|\phi\rangle = e^{i\delta}(\cos\theta|\psi\rangle + \sin\theta|\chi\rangle) .$$

where  $\delta$  is a global phase,  $\theta$  is a “polar angle” in the range  $0 \leq \theta \leq \pi/2$ , and  $|\chi\rangle$  is a normalized vector orthogonal to  $|\psi\rangle$ . We can get rid of the global phase freedom by choosing  $\delta = 0$ , thus working with rays in projective Hilbert-space. A small change in  $|\phi\rangle$  takes the form

$$|d\phi\rangle = d\theta(-\sin\theta|\psi\rangle + \cos\theta|\chi\rangle) + \sin\theta|d\chi\rangle ,$$

which gives

$$\begin{aligned} \langle\phi|d\phi\rangle &= \sin^2\theta\langle\chi|d\chi\rangle , \\ \langle d\phi|d\phi\rangle &= d\theta^2 + \sin^2\theta\langle d\chi|d\chi\rangle . \end{aligned}$$

The resulting line element is

$$\begin{aligned} d\gamma^2 &= d\theta^2 + \sin^2\theta(\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2) \\ &= d\theta^2 + \sin^2\theta(\langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2) , \end{aligned}$$

where  $|d\chi_\perp\rangle = |d\chi\rangle - |\chi\rangle\langle\chi|d\chi\rangle$  is the projection of  $|d\chi\rangle$  orthogonal to  $|\chi\rangle$ .

The line element  $\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2 = \langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2$  is the standard metric on the unit sphere,  $\mathcal{S}_{2D-3}$ , in  $2D - 2$  real dimensions, except that along one real dimension, corresponding to phase changes of  $|\chi\rangle$ , lengths are scaled by a factor  $\cos\theta$ . The  $\sin^2\theta$  in the line element means that lengths along all  $2D - 3$  real dimensions of  $\mathcal{S}_{2D-3}$  are scaled by a factor of  $\sin\theta$ . Thus the integration measure that goes with the Fubini-Study metric is

$$d\Gamma_{|\phi\rangle} = \sin^{2D-3}\theta \cos\theta d\theta dS_{2D-3} ,$$

where  $dS_{2D-3}$  is the standard measure on the unit sphere  $\mathcal{S}_{2D-3}$ . This form of the integration measure is useful for doing integrals over functions of  $|\langle\phi|\psi\rangle| = \cos\theta$ .

In doing this problem, you should never have to calculate an explicit form for the area of a sphere. Instead, the area  $S_{2D-3}$  of  $\mathcal{S}_{2D-3}$  can be left as a normalization constant, whose value cancels out of the ultimate answer to part (c).

- (a) Using the completeness relation, determine the value of the positive constant  $\alpha$ .
- (b) *Find* the value of the integral

$$\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 .$$

(c) One strategy for approximate cloning of an arbitrary state  $|\psi\rangle$  is to measure the isotropic POVM and then make copies of the result  $|\phi\rangle$ . *Find* the average squared fidelity of the copies.