Phys 572 Quantum Information Theory Spring 2017

Homework Problem 8.1

Pretty good measurements. Consider a density operator ρ and a pure-state ensemble decomposition

$$\rho = \sum_{\alpha=1}^{N} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_\alpha |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| \ ,$$

where $|\overline{\psi}_\alpha\rangle = \sqrt{p_\alpha}|\psi_\alpha\rangle$. The support of ρ is the subspace spanned by the eigenvectors with nonzero eigenvalues. The HJW theorem for ensemble decompositions tells us that the decomposition vectors $|\overline{\psi}_\alpha\rangle$ lie in and span the support. In this problem, we restrict attention to the support, which we assume to be D-dimensional, and forget about the rest of Hilbert space. This means that ρ is invertible.

We can define a POVM that has POVM elements

$$E_\alpha = \rho^{-1/2}|\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|\rho^{-1/2} = |\overline{\phi}_\alpha\rangle\langle\overline{\phi}_\alpha| \ ,$$

where $|\overline{\phi}_\alpha\rangle = \rho^{-1/2}|\overline{\psi}_\alpha\rangle = \sqrt{p_\alpha}\rho^{-1/2}|\psi_\alpha\rangle$. The POVM elements are clearly positive operators and the POVM satisfies the completeness relation

$$\sum_\alpha E_\alpha = \rho^{-1/2}\underbrace{\left(\sum_\alpha |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|\right)}_{=\rho}\rho^{-1/2} = I \ .$$

This measurement is called the *pretty good measurement.*

The outcome probabilities $q_\alpha = \mathrm{tr}(\rho E_\alpha)$, when the state is ρ, are the same as the ensemble probabilities p_α. This is the unique property of the pretty good measurement, and it means that the preparation information inequality,

$$H(\mathbf{p}) \ge S(\rho) \ ,$$

and the POVM inequality,

$$H(\mathbf{q}) + \sum_\alpha q_\alpha \log\bigl(\mathrm{tr}(E_\alpha)\bigr) \ge S(\rho) \ ,$$

are constraints on the same probability distribution $\mathbf{p} = \mathbf{q}$.

(a) Which of these inequalities provides the tighter constraint on $H(\mathbf{p})$?

(b) *Show* that the POVM inequality can be rewritten as

$$\sum_\alpha p_\alpha \log\bigl(\langle\psi_\alpha|\rho^{-1}|\psi_\alpha\rangle\bigr) \ge S(\rho) \ .$$

(c) *Show* that

$$H(\mathbf{p}) \ge \sum_\alpha p_\alpha \log\bigl(\langle\psi_\alpha|\rho^{-1}|\psi_\alpha\rangle\bigr) \ .$$