3.8.

(a) The transpose is linear, i.e., $(aA+bB)^T = aA^T+bB^T$, so it is clear that the symmetric and antisymmetric operators form subspaces:

$$\begin{aligned} \text{symmetric operators:}\quad & (aA+bB)^T = aA^T+bB^T = aA+bB\ , \\ \text{antisymmetric operators:}\quad & (aA+bB)^T = aA^T+bB^T = -aA-bB = -(aA+bB)\ . \end{aligned}$$

The matrix elements of symmetric operators are symmetric, i.e., $A_{jk} = A_{kj}$, giving symmetric operators $D(D+1)/2$ independent components. The dimension of the vector space of symmetric operators is thus $D(D+1)/2$. The matrix elements of antisymmetric operators are antisymmetric, i.e., $A_{jk} = -A_{kj}$, giving antisymmetric operators $D(D-1)/2$ independent components.

(b) We can write O as

$$O = \frac{1}{2}(O+O^T) + \frac{1}{2}(O-O^T)\ .$$

The first operator on the right is symmetric, and the second is antisymmetric. This shows that L_V is the *direct sum* of S_V and A_V.

(c) Any pure-state projector $|\psi\rangle\langle\psi|$ on a real vector space is trivially a symmetric operator. The symmetric operators being a subspace, any linear combination of pure-state projectors is also a symmetric operator, so the pure states cannot span all of L_V, since they can never give rise to an antisymmetric operator.

The pure-state projectors do, however, span the space of symmetric operators. The operators $\tau_{jk} = |e_j\rangle\langle e_k|$ form an orthonormal basis for L_V. We can introduce an orthonormal basis for S_V by symmetrizing these operators:

$$\begin{aligned} j = 1,\ldots,D:\quad & \tau_{jj} \\ j<k:\quad & \frac{1}{\sqrt{2}}(\tau_{jk}+\tau_{kj})\ . \end{aligned} \tag{1}$$

Antisymmetrizing these operators gives an orthonormal basis for A_V:

$$j<k:\quad \frac{1}{\sqrt{2}}(\tau_{jk}-\tau_{kj}) \tag{2}$$

A basis of pure-state projectors for S_V is given by

$$\begin{aligned} j = 1,\ldots,D:\quad & |e_j\rangle\langle e_j| = \tau_{jj} \\ j<k:\quad & |\chi_{jk}\rangle\langle\chi_{jk}| = \frac{1}{2}(\tau_{jj}+\tau_{kk}+\tau_{jk}+\tau_{kj})\ , \end{aligned}$$

where

$$|\chi_{jk}\rangle = \frac{1}{\sqrt{2}}(|e_j\rangle + |e_k\rangle)\ .$$

We know this is a basis for S_V because we can write the operators in the symmetric basis of Eq. (1) in terms of these pure-state projectors.

The difference in a complex vector space is that we can also introduce pure-state projectors

$$|\xi_{jk}\rangle\langle\xi_{jk}| = \frac{1}{2}\big(\tau_{jj} + \tau_{kk} - i(\tau_{jk} - \tau_{kj})\big) , \quad j < k,$$

where

$$|\xi_{jk}\rangle = \frac{1}{\sqrt{2}}(|e_j\rangle + i|e_k\rangle) ,$$

thus also recovering the antisymmetric basis operators of Eq. (2).

In a real vector space, the pure-state projectors span the space of symmetric operators, but this is not the entire space of linear operators, since the entire space is the direct sum of the space of symmetric operators and the space of antisymmetric operators. In a complex vector space, the pure-state projectors, when combined with real coefficients, span the real vector space of Hermitian operators. Since the entire space of linear operators is the complexification of the space of Hermitian operators, the pure-state projectors, when combined with complex coefficients, span the entire operator space.

In a complex vector space, there is no way to define symmetric and antisymmetric operators. Suppose A is an operator that is antisymmetric in some basis. Then iA is Hermitian. In its eigenbasis, iA is diagonal and symmetric, which means that A is also diagonal and symmetric in the eigenbasis.

(d) The Pauli operators I, X, and Z are real, symmetric, and orthogonal thus span the 3-dimensional subspace of symmetric operators. The Pauli operator iY is real and antisymmetric and thus spans the 1-dimensional subspace of antisymmetric operators.

(e) The tensor product of two symmetric operators or of two antisymmetric operators is symmetric, and the tensor product of a symmetric and an antisymmetric operator is antisymmetric. For two rebits, the symmetric subspace has $4(4+1)/2 = 10$ dimensions, and the antisymmetric subspace has $4(4-1)/2 = 6$ dimensions. The symmetric Pauli operators for each rebit give 9 orthogonal symmetric product operators: $I \otimes I$, $I \otimes X$, $I \otimes Z$, $X \otimes I$, $X \otimes X$, $X \otimes Z$, $Z \otimes I$, $Z \otimes X$, and $Z \otimes Z$. The tenth orthogonal basis operator is $iY \otimes iY = -Y \otimes Y$. The 6 orthogonal antisymmetric Pauli operators are $iY \otimes I$, $iY \otimes X$, $iY \otimes Z$, $I \otimes iY$, $X \otimes iY$, and $Z \otimes iY$.

A quantum state of two rebits can include a $Y \otimes Y$, local measurements on the two rebits cannot provide any information because their statistics will always involve traces of symmetric rebit operators times iY, and such traces are zero. To determine the quantum state of two rebits from measurement statistics requires using at least one joint measurement.