CMC

Abq

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Lecture 2

Introduction to quantum information

Canadian Summer School on Quantum Information

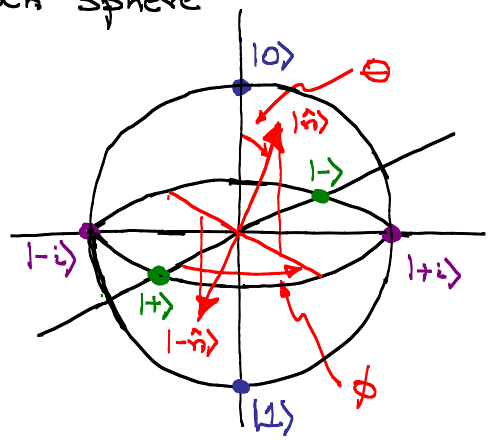
2011 June 6

http://info.phys.unm.edu/~caves/qistutorial/lecture2.pdf

Qubitology

IT: Bloch sphere
P: Implementation of Bloch sphere in physical system.

Bloch sphere



Spin-$\frac{1}{2}$ particle

$|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$

$|+\rangle = |\uparrow_x\rangle$, $|-\rangle = |\downarrow_x\rangle$

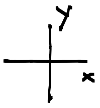
$|+i\rangle = |\uparrow_y\rangle$, $|-i\rangle = |\downarrow_y\rangle$

Polarization of a photon

$|0\rangle = |R\rangle = |\circlearrowright\rangle$, $|1\rangle = |L\rangle = |\circlearrowleft\rangle$

$|+\rangle = |\leftrightarrow\rangle$, $|-\rangle = |\updownarrow\rangle$

$|+i\rangle = |\nearrow\!\!\!\swarrow\rangle$, $|-i\rangle = |\nwarrow\!\!\!\searrow\rangle$



$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \qquad X|\pm\rangle = \pm|\pm\rangle$$

$$|\pm i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i|1\rangle\right) \qquad Y|\pm i\rangle = \pm|\pm i\rangle$$

Two-level atom

$|0\rangle = |e\rangle$, $|1\rangle = |g\rangle$

General state: $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \equiv |\theta,\phi\rangle \equiv |\underset{\substack{\uparrow \\ \text{unit vector}}}{\hat{n}}\rangle$

$$0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$$

$|\hat{n}\rangle$ and $|-\hat{n}\rangle$ are orthogonal.

Pauli operators (Hermitian and unitary)

$$\begin{array}{ll} X = \sigma_x = \sigma_1 & X^2 = Y^2 = Z^2 = I \\ Y = \sigma_y = \sigma_2 & XY = -YX = iZ \\ Z = \sigma_z = \sigma_3 & YZ = -ZY = iX \\ & ZX = -XZ = iY \end{array}$$

$$\left.\begin{array}{l} X^2 = Y^2 = Z^2 = I \\ XY = -YX = iZ \\ YZ = -ZY = iX \\ ZX = -XZ = iY \end{array}\right\} \Longleftrightarrow \sigma_j\sigma_k = \delta_{jk}I + i\epsilon_{jkl}\sigma_l$$

I, X, Y, and Z span the space of operators.

$$\underset{\substack{\uparrow \\ \text{vector operator}}}{\vec{\sigma}}\cdot\hat{n} = Xn_x + Yn_y + Zn_z = \begin{pmatrix}\text{Pauli operator for} \\ \text{direction } \hat{n}\end{pmatrix} = \begin{pmatrix}\text{component of} \\ \vec{\sigma} \text{ along } \hat{n}\end{pmatrix}$$

$$(\vec{\sigma}\cdot\hat{n})^2 = I, \qquad \vec{\sigma}\cdot\hat{n}|\pm\hat{n}\rangle = \pm|\pm\hat{n}\rangle$$

Dynamics: rotations of the Bloch sphere

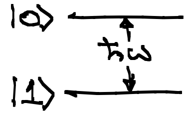
P: Physical systems have Hamiltonians.

Schrödinger equation: $i\hbar \frac{d|\psi\rangle}{dt} = \underset{\substack{\uparrow \\ \text{Hamiltonian}}}{H}|\psi\rangle$

$$|\psi(t)\rangle = \underbrace{e^{-iHt/\hbar}}_{\text{unitary evolution operator}}|\psi(0)\rangle$$

Example: $H = \frac{1}{2}\hbar\omega Z$

$$H|a\rangle = (-1)^a \frac{1}{2}\hbar\omega|a\rangle$$



$$U = e^{-iHt/\hbar} = e^{-iZ\omega t/2}$$

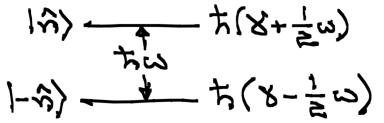
$$U|0\rangle = e^{-i\omega t/2}|0\rangle\ ,\quad U|1\rangle = e^{+i\omega t/2}|1\rangle$$

$$U|\theta,\phi\rangle = e^{-i\omega t/2}\cos(\theta/2)|0\rangle + e^{+i\omega t/2}e^{i\phi}\sin(\theta/2)|1\rangle = e^{-i\omega t/2}|\theta, \underset{\substack{\uparrow \\ \text{rotation} \\ \text{about } z \text{ axis}}}{\phi+\omega t}\rangle$$

General Hamiltonian

$$H = \hbar\gamma I + \frac{1}{2}\hbar\omega\vec{\sigma}\cdot\hat{n} = \hbar\gamma I + \omega\vec{S}\cdot\hat{n}$$

$$\vec{S} = \frac{1}{2}\hbar\vec{\sigma} = \begin{pmatrix}\text{spin-}\frac{1}{2}\text{ angular-momentum} \\ \text{(vector) operator}\end{pmatrix}$$



$$U = e^{-iHt/\hbar} = e^{-i\gamma t}\underbrace{e^{-i\vec{S}\cdot\hat{n}\omega t/\hbar}} = e^{-i\mu}\underbrace{e^{-i\vec{\sigma}\cdot\hat{n}\theta/2}}$$

$\mu = \gamma t$

$\theta = \omega t$

$$U_{\hat{n}}(\theta) = \begin{pmatrix}\text{rotation by angle} \\ \theta \text{ about } \hat{n}\end{pmatrix}$$

Caution! In QIS we tend to use $\gamma = -\frac{1}{2}\omega$ $(\mu = -\theta/2)$, i.e.,

$$U = e^{i\theta/2}e^{-i\vec{\sigma}\cdot\hat{n}\theta/2}$$

What does it mean? We sort of know that

$$U_{\hat{n}}(\theta)|\hat{m}\rangle = (\text{phase})|\underset{\substack{\uparrow \\ \text{rotation operator in three spatial dimensions} \\ \text{for rotation by angle } \theta \text{ about exis } \hat{n}}}{R_{\hat{n}}(\theta)}\hat{m}\rangle$$

Here's a way to know.

$$\vec{\sigma}\cdot R_{\hat{n}}(\theta)\hat{m}\ U_{\hat{n}}(\theta)|\hat{m}\rangle = U_{\hat{n}}(\theta)\underbrace{U_{\hat{n}}^\dagger(\theta)\vec{\sigma}\cdot R_{\hat{n}}(\theta)\hat{m}\ U_{\hat{n}}(\theta)}_{=\vec{\sigma}\cdot\hat{m}\ (\text{from below})}|\hat{m}\rangle = U_{\hat{n}}(\theta)|\hat{m}\rangle$$

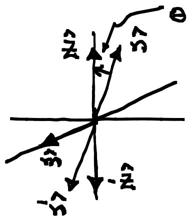
$\Rightarrow$ $U_{\hat{n}}(\theta)|\hat{m}\rangle$ is a normalized $+1$ eigenstate of $\vec{\sigma}\cdot R_{\hat{n}}(\theta)\hat{m}$, so is within a phase of $|R_{\hat{n}}(\theta)\hat{m}\rangle$.

This is stated more compactly as

$$U^{\dagger}_{\tilde{n}}(\theta)\,\vec{\sigma}\cdot\hat{m}\,U_{\tilde{n}}(\theta) = \vec{\sigma}\cdot R^{T}_{\tilde{n}}(\theta)\,\hat{m} = R_{\tilde{n}}(\theta)\,\vec{\sigma}\cdot\vec{m}$$

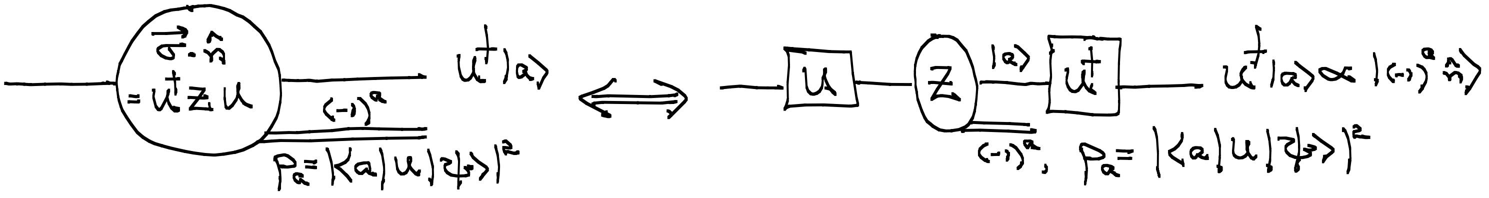
$$\Longleftrightarrow\quad U^{\dagger}_{\tilde{n}}(\theta)\,\vec{\sigma}\,U_{\tilde{n}}(\theta) = R_{\tilde{n}}(\theta)\,\vec{\sigma} \longleftarrow$$ $\vec{\sigma}$ is a vector operator

Measurement: To make a measurment in the basis $|\pm\hat{n}\rangle$, rotate $|\pm\hat{n}\rangle$ to the standard basis $|a\rangle$, and then measure in the standard basis. To get the right post-measurement state, rotate back.



$$R_{\hat{\tilde{n}}}(\theta)\,\hat{n} = \hat{z}$$

$$U^{\dagger}_{\hat{\tilde{n}}}(\theta)\,Z\,U_{\hat{\tilde{n}}}(\theta) = \vec{\sigma}\cdot\hat{n}$$



Other properties of Pauli operators

① $$U_{\hat{n}}(\theta) = \exp\left(-i\vec{\sigma}\cdot\hat{n}\,\theta/2\right) = I\cos(\theta/2) - i\vec{\sigma}\cdot\hat{n}\sin(\theta/2)$$

② $$\sigma_j\sigma_k = \delta_{jk}I + i\epsilon_{jkl}\sigma_l \iff (\vec{\sigma}\cdot\vec{v})(\vec{\sigma}\cdot\vec{w}) = \vec{v}\cdot\vec{w} + i\vec{\sigma}\cdot\vec{v}\times\vec{w}$$

③ $$I = |\hat{n}\rangle\langle\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| = P_{\hat{n}} + P_{-\hat{n}} \quad , \quad P_{\hat{n}} = |\hat{n}\rangle\langle\hat{n}| = \begin{pmatrix}\text{projector}\\ \text{onto } |\hat{n}\rangle\end{pmatrix}$$
$$\vec{\sigma}\cdot\hat{n} = |\hat{n}\rangle\langle\hat{n}| - |-\hat{n}\rangle\langle-\hat{n}| = P_{\hat{n}} - P_{-\hat{n}} \qquad\qquad = \frac{1}{2}(I + \vec{\sigma}\cdot\hat{n})$$

④ $$\mathrm{tr}(\sigma_j) = 0 \,, \quad \mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}$$

⑤ The operators $I = \sigma_0$, $X = \sigma_1$, $Y = \sigma_2$, and $Z = \sigma_3$ span the space of operators, and they are orthogonal, i.e.,

$$\mathrm{tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta} \,, \quad \alpha,\beta = 0,1,2,3.$$

Any operator can be expanded in terms of the σ_α:

$$A = \sum_\alpha A_\alpha\sigma_\alpha = A_0 I + \vec{A}\cdot\vec{\sigma} \,,$$

$$A_\alpha = \frac{1}{2}\mathrm{tr}(\sigma_\alpha A) \,, \qquad \begin{matrix} A_0 = \frac{1}{2}\mathrm{tr}(A) \\ \vec{A} = \frac{1}{2}\mathrm{tr}(\vec{\sigma}A) \,. \end{matrix}$$