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Lecture 3

Introduction to quantum information

Canadian Summer School on Quantum Information

2011 June 6

http://info.phys.unm.edu/~caves/qistutorial/lecture3.pdf

Where we stand. The setting for quantum mechanics is a complex vector space called Hilbert space

States: $|\psi\rangle = \sum_{j=1}^{D} c_j \underset{\text{orthonormal basis}}{|e_j\rangle} \longrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$, $c_j = \langle e_j|\psi\rangle$

ket

$$= \sum_j |e_j\rangle\langle e_j|\psi\rangle$$

$$\langle\psi| = \sum_j c_j^* \langle e_j| \longrightarrow (c_1^* \cdots c_D^*) = \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}^\dagger$$

$$= \sum_j \langle\psi|e_j\rangle\langle e_j|$$

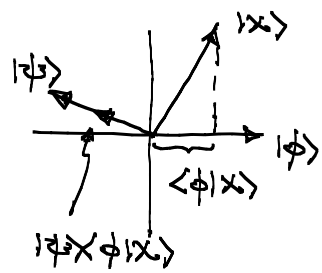
Operators: $A = \sum_{j,k} A_{jk} |e_j\rangle\langle e_k| \longrightarrow \begin{pmatrix} A_{11} & \cdots & A_{1D} \\ \vdots & \ddots & \vdots \\ A_{D1} & \cdots & A_{DD} \end{pmatrix}$

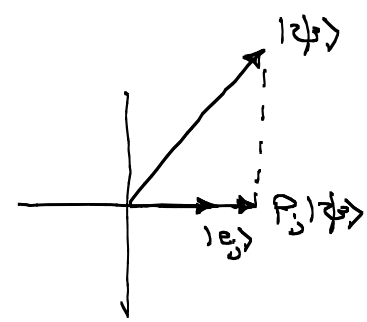
$$= \sum_{j,k} |e_j\rangle\langle e_j|A|e_k\rangle\langle e_k| \qquad A_{jk} = \langle e_j|A|e_k\rangle$$

$$A^\dagger = \sum_{j,k} A_{jk}^* |e_k\rangle\langle e_j| \longrightarrow \begin{pmatrix} A_{11}^* & \cdots & A_{D1}^* \\ \vdots & \ddots & \vdots \\ A_{1D}^* & \cdots & A_{DD}^* \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1D} \\ \vdots & \ddots & \vdots \\ A_{D1} & \cdots & A_{DD} \end{pmatrix}^\dagger$$

$$= \sum_{j,k} |e_k\rangle\langle e_k|A^\dagger|e_j\rangle\langle e_j|$$

$$\big(|\psi\rangle\langle\phi|\big)^\dagger = |\phi\rangle\langle\psi|$$ The adjoint reverses outer products





$$I = \sum_j \underbrace{|e_j\rangle\langle e_j|}_{P_j} = \sum_j P_j$$

one-dimensional projector

$$P_j^2 = P_j$$

true for all projectors

Observable: Hermitian operator $H = \sum_j h_j |e_j\rangle\langle e_j|$, $h_j = h_j^*$
$H = H^\dagger$

Dynamics: Unitary operator $U = e^{-iHt/\hbar} = \sum_j e^{-iE_jt/\hbar} |e_j\rangle\langle e_j|$
$UU^\dagger = I$

Hamiltonian

eigendecompositions

Measurements

In orthonormal basis $|e_j\rangle$

$$p_j = |\langle e_j|\psi\rangle|^2 = \langle e_j|\psi\rangle\langle\psi|e_j\rangle = \langle\psi|e_j\rangle\langle e_j|\psi\rangle = \langle\psi|P_j|\psi\rangle = \mathrm{tr}(|\psi\rangle\langle\psi|P_j)$$

$$\langle H\rangle = \begin{pmatrix}\text{expectation}\\ \text{value of } H\end{pmatrix} = \sum_j p_j h_j = \langle\psi|H|\psi\rangle = \mathrm{tr}(|\psi\rangle\langle\psi|H)$$

Generalizing everything

not necessarily orthogonal

States. Ensemble of states $q_j, |\psi_j\rangle$

ⓐ Measurement in basis $|e_k\rangle$

$$p_k = \sum_j \underbrace{p_{k|j}}_{|\langle e_k|\psi_j\rangle|^2} q_j = \langle e_k|\underbrace{\left(\sum_j q_j|\psi_j\rangle\langle\psi_j|\right)}_{\rho = \text{(density operator)}}|e_k\rangle = \langle e_k|\rho|e_k\rangle = \mathrm{tr}(P_k\rho)$$

__pure__ state (one element in ensemble) vs. __mixed__ state

$$\langle H\rangle = \sum_k p_k h_k = \mathrm{tr}(H\rho)$$

ⓑ Properties of density operator

① $\rho = \rho^\dagger \implies \rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$, $\lambda_j = \lambda_j^* = \begin{pmatrix}\text{probability}\\ \text{for } |e_j\rangle\end{pmatrix}$

eigendecomposition

② $\sum_j \lambda_j = 1 \iff \mathrm{tr}(\rho) = 1$

③ $\lambda_j \geq 0 \iff \rho \geq 0$ (ρ is a positive operator)

④ ρ is pure, i.e., $\rho = |\psi\rangle\langle\psi| \iff \rho^2 = \rho$

c Qubits

$$\rho = \frac{1}{2}(I + \vec{S}\cdot\vec{\sigma}) = \frac{1}{2}(I + S\hat{n}\cdot\vec{\sigma}) \; , \quad \langle\vec{\sigma}\rangle = \mathrm{tr}(\rho\vec{\sigma}) = \vec{S}$$

1 and 2

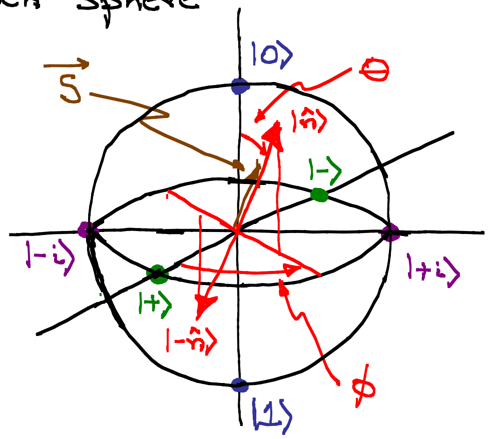
$$\frac{1}{2}(1+S)|\hat{n}\rangle\langle\hat{n}| + \frac{1}{2}(1-S)|-\hat{n}\rangle\langle-\hat{n}|$$

3 $\Rightarrow$ $S \le 1$

Pure state: $S = 1$

Mixed state: $S < 1$

Bloch sphere



Nonuniqueness of ensemble decompositions

d Trace

$$\mathrm{tr}(A) = \sum_j \langle e_j|A|e_j\rangle$$

$$\mathrm{tr}(|\psi\rangle\langle\phi|) = \sum_j \langle e_j|\psi\rangle\langle\phi|e_j\rangle = \sum_j \langle\phi|e_j\rangle\langle e_j|\psi\rangle = \langle\phi|\psi\rangle$$

The trace is a map from operators to complex numbers that takes outer products to inner products.

$$p_j = \mathrm{tr}(P_j\rho) = \langle e_j|\rho|e_j\rangle$$

$$\langle H\rangle = \mathrm{tr}(H\rho)$$

e Tensor product Two systems $|e_j\rangle\otimes|f_k\rangle = |e_j, f_k\rangle$

$$|\psi_{AB}\rangle = \sum_{j,k} |e_j, f_k\rangle \underbrace{\langle e_j, f_k|\psi_{AB}\rangle}_{c_{jk}}$$

Arbitrary linear combinations are what makes this a tensor product.

Product state:

$$|\psi_{AB}\rangle = |\psi_A\rangle\otimes|\psi_B\rangle$$

$$c_{jk} = c_j d_k$$

$$\rho_{AB} = \sum_{\substack{j,k\\ j',k'}} |e_j, f_k\rangle \underbrace{\langle e_j, f_k|\rho_{AB}|e_{j'}, f_{k'}\rangle}_{(\rho_{AB})_{jk,j'k'}} \langle e_{j'}, f_{k'}|$$

Product state:

$$\rho_{AB} = \rho_A\otimes\rho_B$$

$$(\rho_{AB})_{jk,j'k'} = (\rho_A)_{jj'}(\rho_B)_{kk'}$$

Partial inner product:

$$\langle\phi_B|\psi_{AB}\rangle = \sum_{j,k} |e_j\rangle\langle\phi_B|f_k\rangle\langle e_j,f_k|\psi_{AB}\rangle = \sum_j |e_j\rangle\big(\langle e_j|\otimes\langle\phi_B|\big)|\psi_{AB}\rangle$$

Partial trace:

$$p_{jk} = \langle e_j,f_k|\rho_{AB}|e_j,f_k\rangle$$

$$p_j = \sum_k p_{jk} = \langle e_j|\underbrace{\Big(\sum_k \langle f_k|\rho_{AB}|f_k\rangle\Big)}_{tr_B(\rho_{AB}) = \rho_A}|e_j\rangle = tr_A(P_j\rho_A) = tr(P_j\otimes I\rho_{AB})$$

partial trace (tr_B) — marginal density operator (ρ_A)

F

Schmidt decomposition and entanglement (of bipartite pure states)

$$|\psi_{AB}\rangle\langle\psi_{AB}| \longrightarrow \rho_A = \sum_j \lambda_j |e_j\rangle\langle e_j|,\quad \rho_B = \sum_j \lambda_j |f_j\rangle\langle f_j| \longrightarrow |\psi_{AB}\rangle = \sum_j \sqrt{\lambda_j}\,|e_j\rangle\otimes|f_k\rangle$$

Schmidt decomposition

Product state: 1 Schmidt term
Entanglement: more than 1 Schmidt term

Two qubits: Bell states (maximally entangled)

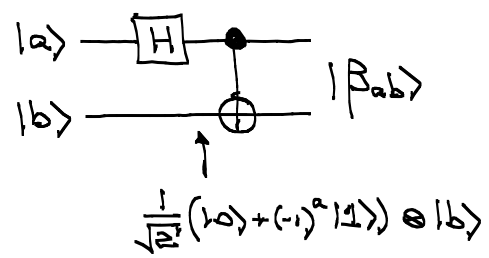
$$|\beta_{ab}\rangle = \frac{1}{\sqrt{2}}\big(|0b\rangle + (-1)^a|1\bar{b}\rangle\big)$$

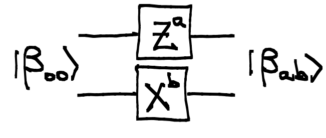
phase bit (a), parity bit (b)

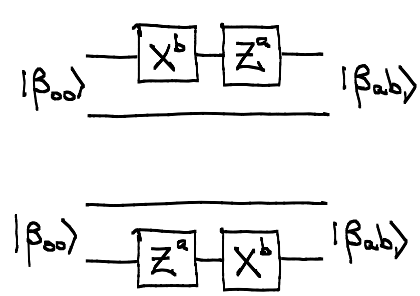
particle interchange

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
symmetric

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
antisymmetric

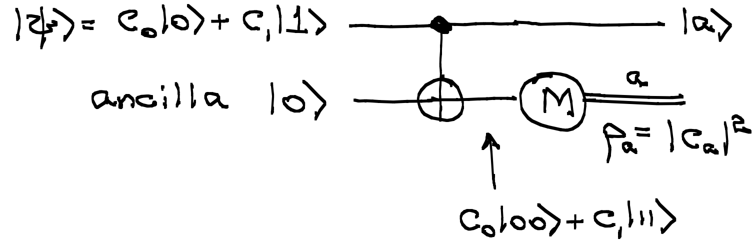






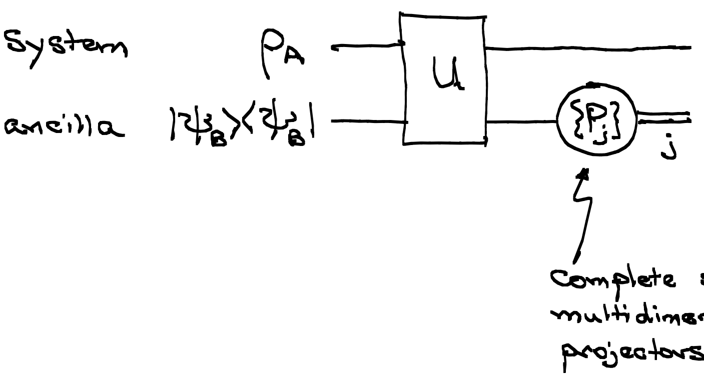
Generalized measurements.

Example: Qubit measurement with ancilla



$${}_2\langle a|\psi\rangle = c_a|a\rangle_1$$

Generalize



quantum operation for outcome j

$$\rho_{A|j} = \frac{\mathrm{tr}_B\left(P_j U \rho_A \otimes |\psi_B\rangle\langle\psi_B| U^\dagger P_j\right)}{p_j} = \frac{\mathcal{A}_j(\rho_A)}{p_j}$$

$$p_j = \mathrm{tr}_A\left(\mathcal{A}_j(\rho_A)\right)$$

A quantum operation is a completely positive, trace-decreasing (or trace-preserving) superoperator

$$P_j = \sum_{k\in S_j} |f_k\rangle\langle f_k|$$

The sets S_j are disjoint sets of basis projectors.

$$\mathrm{tr}_B\left(P_j U \rho_A \otimes |\psi_B\rangle\langle\psi_B| U^\dagger P_j\right) = \sum_{k,k'\in S_j} \mathrm{tr}_B\left(|f_k\rangle\langle f_k| U |\psi_B\rangle \rho_A \langle\psi_B| U^\dagger |f_{k'}\rangle\langle f_{k'}|\right)$$

$$= \sum_{k\in S_j} \underbrace{\langle f_k| U |\psi_B\rangle}_{A_j} \rho_A \underbrace{\langle\psi_B| U^\dagger |f_k\rangle}_{A_j^\dagger}$$

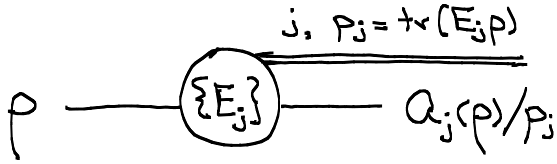
$$\Longrightarrow \quad \mathcal{A}_j(\rho) = \sum_k A_{jk}\, \rho\, A_{jk}^\dagger$$

Kraus operators for operation $\mathcal{A}_j$

$$p_j = \mathrm{tr}\left(\mathcal{A}_j(\rho)\right) = \mathrm{tr}(E_j \rho), \quad E_j = \sum_k A_{jk}^\dagger A_{jk}, \quad E_j = E_j^\dagger \geq 0, \quad \sum_j E_j = I$$

POVM element for outcome j

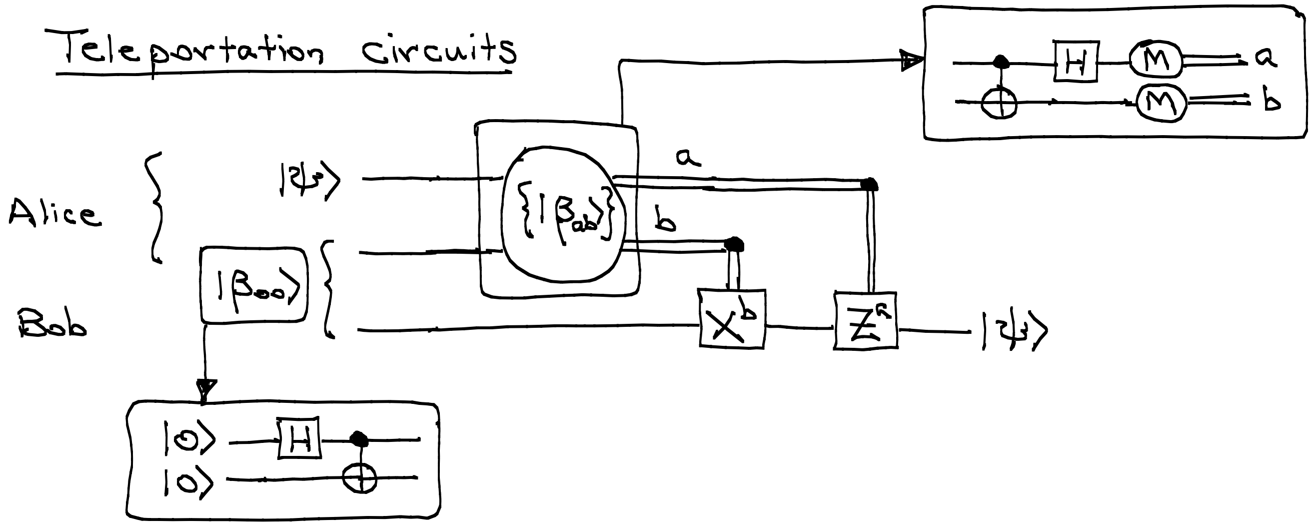
$\{E_j\}$ is called a POVM.

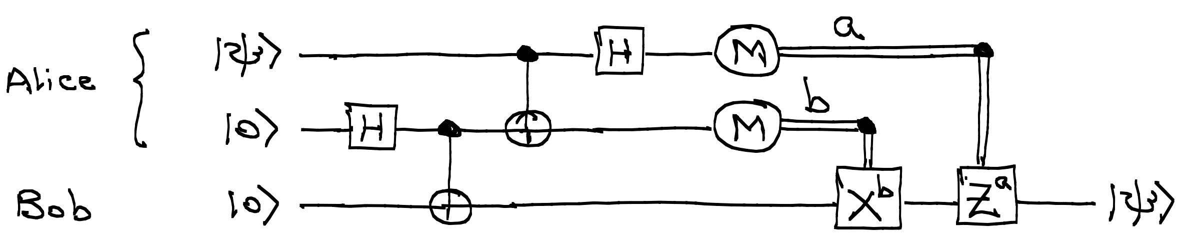


The quantum operation and the POVM are system properties.

The quantum operation gives the measurement statistics and the postmeasurement state. The POVM gives only the measurement statistics.

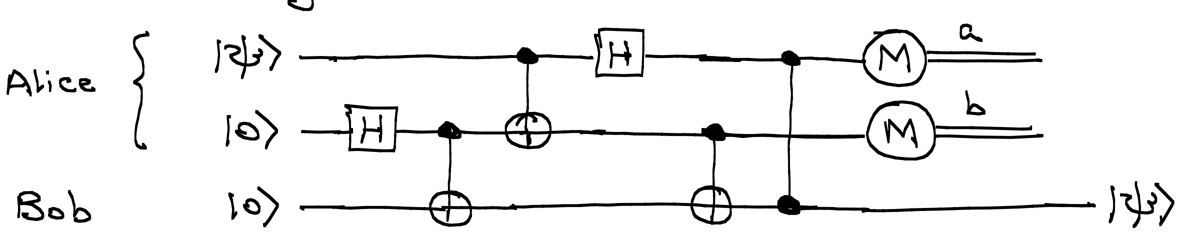
Teleportation circuits





Teleportation circuit

We can refine this circuit in one other way, by using the principle of deferred measurement to move the final controls through the measurement.



Reversible teleportation circuit

The measurements become irrelevant and can be deleted, leaving a reversible teleportation circuit. The measurements tell us after the fact what action was taken by the CNOT and CSIGN gates. Of course, one of the points of teleportation is that Alice and Bob interact only via classical communication (2 bits from Alice to Bob), whereas the reversible circuit has a direct quantum interaction.