Operator formalism and quasidistributions for creation and annihilation operators

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This document is essentially a list of equations. It contains the following sections, each of which begins a new page. Pages are numbered individually by section. Equations are numbered sequentially through the whole document. The document is updated periodically as corrections are made and new results are included.

In the following, A&S refers to M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (U.S. Government Printing Office, Washington, D.C., 1964). The numbers after A&S direct you to particular expressions in Abramowitz and Stegun. As far as I can tell, Wolfram MathWorld tends to use the notation of Abramowitz and Stegun.

1. Basics

$$a = \frac{1}{\sqrt{2}}(x + ip) \qquad x = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad [a, a^\dagger] = 1$$
$$a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \qquad p = -\frac{i}{\sqrt{2}}(a - a^\dagger) \qquad [x, p] = i \tag{1}$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{1}{4}\big|\langle[x,p]\rangle\big|^2 = \frac{1}{4} \tag{2}$$

$$a^\dagger a = \frac{1}{2}(x^2 + p^2 - 1) \qquad a^2 = \frac{1}{2}\Big(x^2 - p^2 + i(xp + px)\Big)$$
$$aa^\dagger = \frac{1}{2}(x^2 + p^2 + 1) \qquad (a^\dagger)^2 = \frac{1}{2}\Big(x^2 - p^2 - i(xp + px)\Big) \tag{3}$$

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \qquad [a^\dagger, a^n] = -na^{n-1} \tag{4}$$

$$[x, p^n] = inp^{n-1} \qquad [p, x^n] = -inx^{n-1} \tag{5}$$

(Use $[A, B^n] = nB^{n-1}[A, B]$ if B commutes with $[A, B]$.)

$$[a, a^\dagger a] = [a, a^\dagger]a = a \tag{6}$$

$$e^{i\theta a^\dagger a} a e^{-i\theta a^\dagger a} = a e^{-i\theta} \qquad e^{\lambda a^\dagger a} a e^{-\lambda a^\dagger a} = a e^{-\lambda}$$
$$e^{i\theta a^\dagger a} a^\dagger e^{-i\theta a^\dagger a} = a^\dagger e^{i\theta} \qquad e^{\lambda a^\dagger a} a^\dagger e^{-\lambda a^\dagger a} = a^\dagger e^{\lambda} \tag{7}$$

(Use $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$)

$$e^{i\theta a^\dagger a} x e^{-i\theta a^\dagger a} = x\cos\theta + p\sin\theta$$
$$e^{i\theta a^\dagger a} p e^{-i\theta a^\dagger a} = -x\sin\theta + p\cos\theta \tag{8}$$

2. Position and momentum bases

$$\begin{aligned} &\langle x'|x\rangle = \delta(x-x') && \langle p'|p\rangle = \delta(p-p') \\ &\int dx\,|x\rangle\langle x| = 1 && \int dp\,|p\rangle\langle p| = 1 \end{aligned} \qquad \langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx} \tag{9}$$

$$\begin{aligned} e^{-ipa}|x\rangle &= |x+a\rangle \qquad \langle x|e^{-ipa} = \langle x-a| \\ e^{ixb}|p\rangle &= |p+b\rangle \qquad \langle p|e^{ixb} = \langle p-b| \end{aligned} \tag{10}$$

$$\langle x|e^{-ipa}|\psi\rangle = \langle x-a|\psi\rangle \qquad \langle p|e^{ixb}|\psi\rangle = \langle p-b|\psi\rangle \tag{11}$$

$$\begin{aligned} \langle x|p|\psi\rangle &= \frac{1}{i}\frac{d}{da}\langle x|e^{ipa}|\psi\rangle\Big|_{a=0} = \frac{1}{i}\frac{d}{da}\langle x+a|\psi\rangle\Big|_{a=0} = \frac{1}{i}\frac{d}{dx}\langle x|\psi\rangle \iff p \leftrightarrow \frac{1}{i}\frac{d}{dx} \\ \langle p|x|\psi\rangle &= i\frac{d}{db}\langle p|e^{-ixb}|\psi\rangle\Big|_{b=0} = i\frac{d}{db}\langle p+b|\psi\rangle\Big|_{b=0} = i\frac{d}{dp}\langle p|\psi\rangle \iff x \leftrightarrow i\frac{d}{dp} \end{aligned} \tag{12}$$

$$a = \frac{1}{\sqrt{2}}(x+ip) \leftrightarrow \begin{cases} \dfrac{1}{\sqrt{2}}\left(x + \dfrac{d}{dx}\right) & \text{(position basis)} \\ \dfrac{i}{\sqrt{2}}\left(\dfrac{d}{dp} + p\right) & \text{(momentum basis)} \end{cases} \tag{13}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x-ip) \leftrightarrow \begin{cases} \dfrac{1}{\sqrt{2}}\left(x - \dfrac{d}{dx}\right) & \text{(position basis)} \\ \dfrac{i}{\sqrt{2}}\left(\dfrac{d}{dp} - p\right) & \text{(momentum basis)} \end{cases} \tag{14}$$

3. Displacement operator

$$D(a,\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} = e^{i(\alpha_2 x - \alpha_1 p)} = e^{-i\alpha_1\alpha_2/2} e^{i\alpha_2 x} e^{-i\alpha_1 p} = e^{i\alpha_1\alpha_2/2} e^{-i\alpha_1 p} e^{i\alpha_2 x} \tag{15}$$

$$\alpha = \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) \qquad \begin{aligned} \alpha_1 &= \frac{1}{\sqrt{2}}(\alpha + \alpha^*) \\ \alpha_2 &= -\frac{i}{\sqrt{2}}(\alpha - \alpha^*) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_1} - i\frac{\partial}{\partial\alpha_2}\right) \qquad & \frac{\partial}{\partial\alpha_1} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right) \\ \frac{\partial}{\partial\alpha^*} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_1} + i\frac{\partial}{\partial\alpha_2}\right) \qquad & \frac{\partial}{\partial\alpha_2} &= \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\alpha^*}\right) \end{aligned} \tag{17}$$

$$\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\alpha_2^2}\right) \tag{18}$$

$$D^{-1}(a,\alpha) = D^\dagger(a,\alpha) = D(a,-\alpha) = D(-a,\alpha) \tag{19}$$

$$\begin{aligned} D(a,\alpha)|x\rangle &= e^{i\alpha_1\alpha_2/2} e^{i\alpha_2 x}|x+\alpha_1\rangle \qquad & \langle x|D(a,\alpha) &= e^{-i\alpha_1\alpha_2/2} e^{i\alpha_2 x}\langle x-\alpha_1| \\ D(a,\alpha)|p\rangle &= e^{-i\alpha_1\alpha_2/2} e^{-i\alpha_1 p}|p+\alpha_2\rangle \qquad & \langle p|D(a,\alpha) &= e^{i\alpha_1\alpha_2/2} e^{-i\alpha_1 p}\langle p-\alpha_2| \end{aligned} \tag{20}$$

$$\begin{aligned} \langle x|D(a,\alpha)|x'\rangle &= e^{-i\alpha_1\alpha_2/2} e^{i\alpha_2 x}\delta(x - x' - \alpha_1) \\ \langle p|D(a,\alpha)|p'\rangle &= e^{i\alpha_1\alpha_2/2} e^{-i\alpha_1 p}\delta(p - p' - \alpha_2) \end{aligned} \tag{21}$$

$$D(\alpha,\beta) = e^{\beta\alpha^* - \beta^*\alpha} = e^{2i(\beta_I\alpha_R - \beta_R\alpha_I)} = e^{i(\beta_2\alpha_1 - \beta_1\alpha_2)} \tag{22}$$

$$D(\alpha,\alpha) = 1 \qquad D(\alpha, r\alpha) = 1,\ r \text{ real} \tag{23}$$

$$D^*(\alpha,\beta) = D(\alpha^*,\beta^*) = D(\alpha,-\beta) = D(-\alpha,\beta) = D(\beta,\alpha) \tag{24}$$

$$D(a,\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger} \tag{25}$$

(Use BCH: $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if A and B commute with $[A,B]$.)

$$D^\dagger(a,\alpha)aD(a,\alpha) = a + \alpha \tag{26}$$

$$D^\dagger(a,\alpha)D(a,\beta)D(a,\alpha) = D(a+\alpha,\beta) = D(\alpha,\beta)D(a,\beta) \tag{27}$$

$$\begin{aligned} D(a,\alpha)D(a,\beta) &= e^{(\alpha\beta^* - \alpha^*\beta)/2}D(a,\alpha+\beta) \\ &= D(\beta,\alpha/2)D(a,\alpha+\beta) \\ &= D(\beta,\alpha)D(a,\beta)D(a,\alpha) \end{aligned} \tag{28}$$

$$\begin{aligned} &D(a,\alpha)D(a,\beta) = D(a,\beta)D(a,\alpha) \quad \Longleftrightarrow \quad D(\beta,\alpha) = 1 \\ \Longleftrightarrow \quad &\begin{pmatrix}\text{area subtended by} \\ \alpha \text{ and } \beta\end{pmatrix} = \frac{1}{2i}(\alpha\beta^* - \alpha^*\beta) = \alpha_I\beta_R - \alpha_R\beta_I = \pi k \\ &\begin{pmatrix}\text{area subtended by} \\ (\alpha_1,\alpha_2) \text{ and } (\beta_1,\beta_2)\end{pmatrix} = \alpha_2\beta_1 - \alpha_1\beta_2 = 2\pi k \end{aligned} \tag{29}$$

$$D(\beta,\alpha/2) - D^*(\beta,\alpha/2) = 2i\sin\!\left(i\,\frac{\beta\alpha^* - \beta^*\alpha}{2}\right) = -2i\sin\!\left(i\,\frac{\alpha\beta^* - \alpha^*\beta}{2}\right) \tag{30}$$

$$[D(a,\alpha), D(a,\beta)] = [D(\beta,\alpha/2) - D^*(\beta,\alpha/2)]D(a,\alpha+\beta) = -2i\sin\!\left(i\,\frac{\alpha\beta^* - \alpha^*\beta}{2}\right)D(a,\alpha+\beta) \tag{31}$$

$$\begin{aligned}
D(a,\alpha)D^\dagger(a,\beta) &= e^{-(\alpha\beta^*-\alpha^*\beta)/2}D(a,\alpha-\beta) \\
&= D(\beta,-\alpha/2)D(a,\alpha-\beta) \\
&= D(\beta,-\alpha)D^\dagger(a,\beta)D(a,\alpha)
\end{aligned} \tag{32}$$

$$D^\dagger(a,\beta)D(a,\alpha) = e^{(\alpha\beta^*-\alpha^*\beta)/2}D(a,\alpha-\beta) = D(\beta,\alpha/2)D(a,\alpha-\beta) \tag{33}$$

$$e^{i\theta a^\dagger a}D(a,\alpha)e^{-i\theta a^\dagger a} = D(ae^{-i\theta},\alpha) = D(a,\alpha e^{i\theta}) \tag{34}$$

$$\begin{aligned}
e^{\lambda a^\dagger a}D(a,\alpha)e^{-\lambda a^\dagger a} &= e^{-|\alpha|^2/2}e^{\lambda a^\dagger a}e^{\alpha a^\dagger}e^{-\lambda a^\dagger a}e^{\lambda a^\dagger a}e^{-\alpha^* a}e^{-\lambda a^\dagger a} \\
&= e^{-|\alpha|^2/2}e^{\alpha a^\dagger e^\lambda}e^{-\alpha^* a e^{-\lambda}} \\
&= e^{-|\alpha|^2/2}e^{\alpha e^\lambda a^\dagger}e^{-\alpha^* e^\lambda a}e^{2\alpha^* a\sinh\lambda} \\
&= e^{-|\alpha|^2(1-e^{2\lambda})/2}D(a,\alpha e^\lambda)e^{2\alpha^* a\sinh\lambda}
\end{aligned} \tag{35}$$

4. Number states

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{36}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad\qquad a|n\rangle = \sqrt{n}|n-1\rangle \tag{37}$$

$$a^\dagger a|n\rangle = n|n\rangle \tag{38}$$

$$\langle n|m\rangle = \delta_{nm} \tag{39}$$

$$\langle n|a|n\rangle = 0 \qquad \begin{array}{c}\langle n|a^2|n\rangle = 0\\ \langle n|a^\dagger a|n\rangle = n\end{array} \tag{40}$$

$$\begin{array}{ll}\langle n|x|n\rangle = 0 & \langle n|x^2|n\rangle = \langle n|p^2|n\rangle = n+\dfrac{1}{2}\\ \langle n|p|n\rangle = 0 & \langle n|(xp+px)|n\rangle = 0\end{array} \tag{41}$$

$$\begin{aligned} e^{-\alpha^* a}|n\rangle &= \sum_{k=0}^{\infty}\frac{(-\alpha^*)^k}{k!}a^k|n\rangle = \sum_{k=0}^{n}\frac{(-\alpha^*)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle \\ \langle n|e^{\alpha a^\dagger} &= \sum_{k=0}^{n}\frac{\alpha^k}{k!}\sqrt{\frac{n!}{(n-k)!}}\langle n-k| \end{aligned} \tag{42}$$

$$\begin{aligned} m\ge n: \quad \langle m|e^{\alpha a^\dagger}e^{-\alpha^* a}|n\rangle &= \sum_{l=0}^{m}\sum_{k=0}^{n}\frac{\alpha^l(-\alpha^*)^k}{l!\,k!}\sqrt{\frac{m!\,n!}{(m-l)!\,(n-k)!}}\langle m-l|n-k\rangle \\ &= \sum_{k=0}^{n}\frac{\alpha^{m-n}(-|\alpha|^2)^k}{(m-n+k)!\,k!}\frac{\sqrt{m!\,n!}}{(n-k)!} \qquad \text{(Here we use } m\ge n.\text{)} \\ &= \sqrt{\frac{n!}{m!}}\,\alpha^{m-n}\sum_{k=0}^{n}\frac{(n+m-n)!}{k!\,(n-k)!\,(m-n+k)!}(-|\alpha|^2)^k \\ &= \sqrt{\frac{n!}{m!}}\,\alpha^{m-n}L_n^{(m-n)}\big(|\alpha|^2\big) \end{aligned} \tag{43}$$

[$L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial of A&S 22.3.9.]

$$\langle m|D(a,\alpha)|n\rangle = \begin{cases} \sqrt{\dfrac{n!}{m!}}\,e^{-|\alpha|^2/2}\alpha^{m-n}L_n^{(m-n)}\big(|\alpha|^2\big)\,, & m\ge n \\ \sqrt{\dfrac{m!}{n!}}\,e^{-|\alpha|^2/2}(-\alpha^*)^{n-m}L_m^{(n-m)}\big(|\alpha|^2\big)\,, & m\le n \end{cases} \tag{44}$$

$$\langle m|D(a,\alpha^*)|n\rangle = (-1)^{m-n}\langle n|D(a,\alpha)|m\rangle \tag{45}$$

$$\langle n|D(a,\alpha)|n\rangle = e^{-|\alpha|^2/2}L_n\big(|\alpha|^2\big) \tag{46}$$

$$e^{-i\theta a^\dagger a}|n\rangle = e^{-in\theta}|n\rangle \tag{47}$$

$$0 = \langle x|a|0\rangle = \frac{1}{\sqrt{2}}\left(x+\frac{d}{dx}\right)\langle x|0\rangle \qquad \Longrightarrow \qquad \langle x|0\rangle = \frac{1}{\pi^{1/4}}e^{-x^2/2} \quad \text{(phase chosen by convention)} \tag{48}$$

$$\langle p|0\rangle = \int_{-\infty}^{\infty}dx\,\langle p|x\rangle\langle x|0\rangle = \frac{1}{\pi^{1/4}}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,e^{-ipx}e^{-x^2/2} = \frac{1}{\pi^{1/4}}e^{-p^2/2}$$

$$
\begin{aligned}
\langle x|n\rangle &= \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|0\rangle \\
&= \frac{1}{\sqrt{2^n n!}}\langle x|(x-ip)^n|0\rangle \\
&= \frac{1}{\sqrt{2^n n!}}\left(x-\frac{d}{dx}\right)^n\langle x|0\rangle \\
&= \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\underbrace{\left(x-\frac{d}{dx}\right)^n e^{-x^2/2}}_{=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}} \\
&= \frac{e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}}\underbrace{(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}}_{=H_n(x)}
\end{aligned}
\tag{49}
$$

[Use Rodrigues's formula for the Hermite polynomial $H_n(x)$: A&S 22.11.7.]

$$
\langle x|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}e^{-x^2/2}H_n(x) = \frac{1}{\sqrt{2^n n!}}H_n(x)\langle x|0\rangle
\tag{50}
$$

5. Normal ordering

Normal ordering, denoted by paired colons, applies to functions of creation and annihilation operators, i.e., expressions written in terms of a and $a^\dagger$. It means to move all annihilation operators to the right and all creation operators to the left without regard to commutators. It is meaningless to refer to the normal-ordered form of an operator A, i.e., to write $\mathbf{:}A\mathbf{:}$, because the result of normal ordering depends on how A is written in terms of creation and annihilation operators. For example, if $A = aa^\dagger = a^\dagger a + 1$, the result of normal ordering the first form is $a^\dagger a$, but the result of normal ordering the second form is $a^\dagger a + 1$.

$$
\begin{aligned}
\mathbf{:}(a^\dagger a)^k\mathbf{:} &\equiv (a^\dagger)^k a^k \\
&= \sum_{n,m=0}^{\infty} |n\rangle\langle n|(a^\dagger)^k a^k|m\rangle\langle m| \\
&= \sum_{n=k}^{\infty} |n\rangle\langle n|(a^\dagger)^k a^k|n\rangle\langle n| \\
&= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} a^\dagger a(a^\dagger a-1)\cdots(a^\dagger a-k+1)|n\rangle\langle n| \\
&= a^\dagger a(a^\dagger a-1)\cdots(a^\dagger a-k+1) \equiv (a^\dagger a)^{(k)} = (-1)^k(-a^\dagger a)_k
\end{aligned}
\tag{51}
$$

The final form in Eq. (51) is called a *falling factorial*. Its expectation value is a *factorial moment*. The notation comes from the *Pochhammer symbol*: $(x)_k \equiv x(x+1)\cdots(x+k-1) = (x+k)!/x!$, which is the rising factorial. The falling factorial, $(-1)^k(-x)_k = x(x-1)\cdots(x-k+1)$, which is what we have here, is also, confusingly, sometimes denoted by the Pochhammer symbol.

$$
\begin{aligned}
\mathbf{:}f(a^\dagger a)\mathbf{:} &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\mathbf{:}(a^\dagger a)^k\mathbf{:} \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n|\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}f^{(k)}(0)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\mathbf{:}e^{-\lambda a^\dagger a}\mathbf{:} &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}(a^\dagger)^k a^k \\
&= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n|\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}(-\lambda)^k \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n|(1-\lambda)^n \\
&= (1-\lambda)^{a^\dagger a} \\
&= e^{\ln(1-\lambda)a^\dagger a}
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
{:}e^{-\lambda(a^\dagger-\alpha^*)(a-\alpha)}{:} &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}(a^\dagger-\alpha^*)^k(a-\alpha)^k \\
&= D(a,\alpha)\left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!}(a^\dagger)^k a^k\right) D^\dagger(a,\alpha) \\
&= D(a,\alpha){:}e^{-\lambda a^\dagger a}{:}D^\dagger(a,\alpha) \\
&= (1-\lambda)^{(a^\dagger-\alpha^*)(a-\alpha)}
\end{aligned} \tag{54}
$$

6. Antinormal ordering

Antinormal ordering, denoted here, for lack of thinking of anything better, by paired bullets, applies to functions of creation and annihilation operators, i.e., expressions written in terms of a and $a^\dagger$. It means to move all annihilation operators to the left and all creation operators to the right without regard to commutators.

$$\begin{aligned}
\bullet(a^\dagger a)^k\bullet &\equiv a^k(a^\dagger)^k \\
&= \sum_{n,m=0}^{\infty} |n\rangle\langle n|a^k(a^\dagger)^k|m\rangle\langle m| \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n|a^k(a^\dagger)^k|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} \frac{(n+k)!}{n!}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+k)|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} aa^\dagger(aa^\dagger+1)\cdots(aa^\dagger+k-1)|n\rangle\langle n| \\
&= aa^\dagger(aa^\dagger+1)\cdots(aa^\dagger+k-1) \equiv (aa^\dagger)_k
\end{aligned} \tag{55}$$

The final form in Eq. (55) is called a *rising factorial.* The notation comes from the *Pochhammer symbol*: $(x)_k \equiv x(x+1)\cdots(x+k-1) = (x+k)!/x!$, which is the rising factorial.

$$\begin{aligned}
\bullet f(a^\dagger a)\bullet &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\bullet(a^\dagger a)^k\bullet \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\sum_{k=0}^{\infty}\frac{(n+k)!}{k!\,n!}f^{(k)}(0)
\end{aligned} \tag{56}$$

$$\begin{aligned}
\bullet e^{-\lambda aa^\dagger}\bullet &= \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}a^k(a^\dagger)^k \\
&= \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+k)|n\rangle\langle n| \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\sum_{k=0}^{\infty}\frac{(n+1)(n+2)\cdots(n+k)}{k!}(-\lambda)^k \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\sum_{k=0}^{\infty}\frac{[-(n+1)][-(n+1)-1]\cdots[-(n+1)-k+1]}{k!}\lambda^k \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|(1+\lambda)^{-(n+1)} \\
&= (1+\lambda)^{-aa^\dagger} \\
&= e^{-\ln(1+\lambda)aa^\dagger}
\end{aligned} \tag{57}$$

$$
\begin{aligned}
\bullet e^{-\lambda(a^\dagger-\alpha^*)(a-\alpha)}\bullet &= \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}(a^\dagger-\alpha^*)^k(a-\alpha)^k \\
&= D(a,\alpha)\left(\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}(a^\dagger)^k a^k\right)D^\dagger(a,\alpha) \\
&= D(a,\alpha)\bullet e^{-\lambda a^\dagger a}\bullet D^\dagger(a,\alpha) \\
&= (1+\lambda)^{-(a-\alpha)(a^\dagger-\alpha^*)}
\end{aligned}
\tag{58}
$$

7. Coherent states

$$|\alpha\rangle \equiv D(a,\alpha)|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{59}$$

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{60}$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle\alpha|a^\dagger = \langle\alpha|\alpha^* \tag{61}$$

$$e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-i\theta a^\dagger a}D(a,\alpha)|0\rangle = D(a,\alpha e^{-i\theta})e^{-i\theta a^\dagger a}|0\rangle = |\alpha e^{-i\theta}\rangle \tag{62}$$

$$\langle\alpha|a|\alpha\rangle = \alpha \qquad \begin{aligned}\langle\alpha|a^2|\alpha\rangle &= \alpha^2\\ \langle\alpha|a^\dagger a|\alpha\rangle &= |\alpha|^2\\ \langle\alpha|(a^\dagger a)^2|\alpha\rangle &= |\alpha|^4+|\alpha|^2\\ \langle\alpha|(\Delta a^\dagger a)^2|\alpha\rangle &= |\alpha|^2\end{aligned} \tag{63}$$

$$\begin{aligned}\langle\alpha|x|\alpha\rangle &= \alpha_1 \qquad & \langle\alpha|(\Delta x)^2|\alpha\rangle = \langle\alpha|(\Delta p)^2|\alpha\rangle = \frac{1}{2}\\ \langle\alpha|p|\alpha\rangle &= \alpha_2 \qquad & \langle\alpha|(\Delta x\Delta p+\Delta p\Delta x)|\alpha\rangle = 0\end{aligned} \tag{64}$$

$$\begin{aligned}\langle x|\alpha\rangle &= \langle x|D(a,\alpha)|0\rangle = e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}\langle x-\alpha_1|0\rangle = \frac{e^{-i\alpha_1\alpha_2/2}}{\pi^{1/4}}e^{-(x-\alpha_1)^2/2}e^{i\alpha_2 x}\\ \langle p|\alpha\rangle &= \langle p|D(a,\alpha)|0\rangle = e^{i\alpha_1\alpha_2/2}e^{-i\alpha_1 p}\langle p-\alpha_2|0\rangle = \frac{e^{i\alpha_1\alpha_2/2}}{\pi^{1/4}}e^{-(p-\alpha_2)^2/2}e^{-i\alpha_1 p}\end{aligned} \tag{65}$$

$$\langle x|\alpha\rangle = \langle -x|-\alpha\rangle \qquad \langle p|\alpha\rangle = \langle -p|-\alpha\rangle \tag{66}$$

$$\langle 0|D(a,\alpha)|0\rangle = \langle 0|\alpha\rangle = e^{-|\alpha|^2/2} \tag{67}$$

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(a,\beta)D(a,\alpha)|0\rangle = D(\beta,\alpha/2)e^{-|\alpha-\beta|^2/2} = e^{-|\alpha|^2/2}e^{-|\beta|^2/2}e^{\alpha\beta^*} \tag{68}$$

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{69}$$

$$\begin{aligned}D(a,\alpha)|\beta\rangle &= D(a,\alpha)D(a,\beta)|0\rangle = D(\beta,\alpha/2)|\beta+\alpha\rangle\\ \langle\beta|D(a,\alpha) &= \langle 0|D^\dagger(a,\beta)D(a,\alpha) = \langle\beta-\alpha|D(\beta,\alpha/2)\end{aligned} \tag{70}$$

$$\langle\beta|D(a,\alpha)|\beta\rangle = e^{-|\alpha|^2/2}\langle\beta|e^{\alpha a^\dagger}e^{-\alpha^* a}|\beta\rangle = e^{-|\alpha|^2/2}D(\beta,\alpha) \tag{71}$$

$$\begin{aligned}\langle\gamma|D(a,\alpha)|\beta\rangle &= e^{-|\alpha|^2/2}\langle\gamma|e^{\alpha a^\dagger}e^{-\alpha^* a}|\beta\rangle\\ &= e^{-|\alpha|^2/2}e^{\alpha\gamma^*-\alpha^*\beta}\langle\gamma|\beta\rangle\\ &= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}e^{-|\gamma|^2/2}e^{\alpha\gamma^*-\alpha^*\beta+\beta\gamma^*}\end{aligned} \tag{72}$$

$$\langle\gamma|D(a,\alpha)|\beta\rangle = D(\gamma,\alpha/2)D(\beta,\alpha/2)D(\gamma,\beta/2)e^{-|\alpha+\beta-\gamma|^2/2} \tag{73}$$

$$e^{-\lambda a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-\lambda})^n}{\sqrt{n!}}|n\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2}|\alpha e^{-\lambda}\rangle \tag{74}$$

$$e^{-\lambda a^\dagger a}|\alpha\rangle = e^{-\lambda a^\dagger a}D(a,\alpha)|0\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2}D(a,\alpha e^{-\lambda})|0\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2}|\alpha e^{-\lambda}\rangle \tag{75}$$

$$\langle\alpha|e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2(1-e^{-i\theta})} \tag{76}$$

$$\langle\alpha|e^{-\lambda a^\dagger a}|\alpha\rangle = \langle\alpha|{:}e^{-(1-e^{-\lambda})a^\dagger a}{:}|\alpha\rangle = e^{-(1-e^{-\lambda})|\alpha|^2} \tag{77}$$

$$\begin{aligned}
\alpha &= \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) = |\alpha|e^{i\phi} \\
d^2\alpha &= d\alpha_R\, d\alpha_I = \frac{d\alpha_1\, d\alpha_2}{2} = |\alpha| d|\alpha|\, d\phi = \frac{1}{2} d|\alpha|^2\, d\phi \\
\delta^2(\alpha) &= \delta(\alpha_R)\delta(\alpha_I) = 2\delta(\alpha_1)\delta(\alpha_2)
\end{aligned} \tag{78}$$

$$\begin{aligned}
\Big\langle n \Big| \Big(\int d^2\alpha\, |\alpha\rangle\langle\alpha| \Big) \Big| m \Big\rangle &= \int d^2\alpha\, \langle n|\alpha\rangle\langle\alpha|m\rangle \\
&= \frac{1}{\sqrt{n!\, m!}} \int d^2\alpha\, e^{-|\alpha|^2} \alpha^n (\alpha^*)^m \\
&= \frac{1}{2\sqrt{n!\, m!}} \int d|\alpha|^2\, d\phi\, e^{-|\alpha|^2} |\alpha|^{n+m} e^{i(n-m)\phi} \\
&= \frac{\pi}{n!}\delta_{nm} \int_0^\infty du\, e^{-u} u^n \\
&= \pi\delta_{nm}
\end{aligned} \tag{79}$$

$$1 = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi}\, D(a,\alpha)|0\rangle\langle 0|D^\dagger(a,\alpha) \tag{80}$$

$$\mathrm{tr}(A) = \int \frac{d^2\alpha}{\pi}\, \langle\alpha|A|\alpha\rangle = \int \frac{d^2\alpha}{\pi}\, \langle 0|D^\dagger(a,\alpha) A D(a,\alpha)|0\rangle \tag{81}$$

$$\begin{aligned}
\mathcal{I} = 1 \odot 1 &= \sum_{n,m} |n\rangle\langle n| \odot |m\rangle\langle m| = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi}\, |\alpha\rangle\langle\alpha| \odot |\beta\rangle\langle\beta| \\
\mathbf{I} = \mathcal{I}^\# &= \sum_{n,m} |n\rangle\langle m| \odot |m\rangle\langle n| = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi}\, |\alpha\rangle\langle\beta| \odot |\beta\rangle\langle\alpha|
\end{aligned} \tag{82}$$

$$\begin{aligned}
\mathcal{I}(A) &= A\,, \qquad \mathcal{I}|A) = 1\mathrm{tr}(A) \\
\mathbf{I}(A) &= 1\mathrm{tr}(A)\,, \qquad \mathbf{I}|A) = A
\end{aligned} \tag{83}$$

8. Parity

$$\begin{aligned} P^\dagger x P &= -x \\ P^\dagger p P &= -p \end{aligned} \quad \iff \quad P^\dagger a P = -a \tag{84}$$

$$aP|0\rangle = -Pa|0\rangle = 0 \quad \Longrightarrow \quad P|0\rangle = e^{i\delta}|0\rangle = |0\rangle \quad \text{(choose phase } \delta = 0) \tag{85}$$

$$P|n\rangle = \frac{1}{\sqrt{n!}} P(a^\dagger)^n|0\rangle = \frac{(-1)^n}{\sqrt{n!}}(a^\dagger)^n P|0\rangle = (-1)^n|n\rangle \tag{86}$$

$$P = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n| = (-1)^{a^\dagger a} = P^\dagger \tag{87}$$

$$PD(a,\alpha)P = D(a,-\alpha) = D^\dagger(a,\alpha) \tag{88}$$

$$P|\alpha\rangle = PD(a,\alpha)|0\rangle = D(a,-\alpha)P|0\rangle = D(a,-\alpha)|0\rangle = |-\alpha\rangle \tag{89}$$

$$P = \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle\alpha| \tag{90}$$

$$\begin{aligned} P|x\rangle &= \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha|x\rangle = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle-\alpha|-x\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|-x\rangle = |-x\rangle \\ P|p\rangle &= \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha|p\rangle = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle-\alpha|-p\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|-p\rangle = |-p\rangle \end{aligned} \tag{91}$$

$$\begin{aligned} P &= \int dx\, P|x\rangle\langle x| = \int dx\, |-x\rangle\langle x| \\ P &= \int dp\, P|p\rangle\langle p| = \int dp\, |-p\rangle\langle p| \end{aligned} \tag{92}$$

$$\begin{aligned} \operatorname{tr}\big(PD(a,\alpha)\big) &= \int \frac{d^2\beta}{\pi}\, \langle\beta|PD(a,\alpha)|\beta\rangle \\ &= \int \frac{d^2\beta}{\pi}\, \langle-\beta|D(a,\alpha)|\beta\rangle \\ &= e^{-|\alpha|^2/2}\int \frac{d^2\beta}{\pi}\, e^{-\alpha\beta^*-\alpha^*\beta}\langle-\beta|\beta\rangle \\ &= e^{-|\alpha|^2/2}\underbrace{\int \frac{d^2\beta}{\pi}\, e^{-2|\beta|^2}e^{-\beta\alpha^*-\beta^*\alpha}}_{=\frac{1}{2}e^{|\alpha|^2/2}} \\ &= \frac{1}{2} \end{aligned} \tag{93}$$

9. Fourier transform pairs

$$\int \frac{d^2\beta}{\pi}\, D(\beta,\alpha) = \int \frac{d^2\beta}{\pi}\, e^{\alpha\beta^* - \alpha^*\beta} = \int \frac{d\beta_1\, d\beta_2}{2\pi}\, e^{i(\alpha_2\beta_1 - \alpha_1\beta_2)} = 2\pi\delta(\alpha_2)\delta(\alpha_1) = \pi\delta^2(\alpha) \tag{94}$$

$$f(\alpha) = \int \frac{d^2\beta}{\pi}\, \tilde{f}(\beta) D(\beta,\alpha) \qquad\qquad \tilde{f}(\beta) = \int \frac{d^2\alpha}{\pi}\, f(\alpha) D(\alpha,\beta) \tag{95}$$

$$g(\alpha) = f^*(\alpha) \qquad \Longleftrightarrow \qquad \tilde{g}(\beta) = \tilde{f}^*(-\beta) \tag{96}$$

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial \beta} &= \int \frac{d^2\alpha}{\pi}\, \alpha^* f(\alpha) D(\alpha,\beta) \\ \frac{\partial \tilde{f}}{\partial \beta^*} &= -\int \frac{d^2\alpha}{\pi}\, \alpha f(\alpha) D(\alpha,\beta) \end{aligned} \qquad \frac{\partial^2 \tilde{f}}{\partial\beta\,\partial\beta^*} = -\int \frac{d^2\alpha}{\pi}\, |\alpha|^2 f(\alpha) D(\alpha,\beta) \tag{97}$$

$$\begin{aligned} &\int \frac{d^2\alpha}{\pi}\, f(\alpha) g(\alpha) D(\alpha,\beta) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(\beta-\gamma) \\ &\int \frac{d^2\beta}{\pi} \left(\int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(\beta-\gamma) \right) D(\beta,\alpha) = f(\alpha) g(\alpha) \end{aligned} \tag{98}$$

$$\int \frac{d^2\alpha}{\pi}\, f(\alpha) g(\alpha) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(-\gamma) \tag{99}$$

$$\int \frac{d^2\alpha}{\pi}\, |f(\alpha)|^2 D(\alpha,\beta) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{f}^*(\gamma-\beta) \qquad \text{(Parseval's relation)} \tag{100}$$

$$\int \frac{d^2\alpha}{\pi}\, |f(\alpha)|^2 = \int \frac{d^2\beta}{\pi}\, |\tilde{f}(\beta)|^2 \tag{101}$$

10. Gaussian integrals

$$\int_{-\infty}^{\infty} du\, e^{-ax^2} e^{bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \tag{102}$$

$$\int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} D(\alpha,\beta) = \int \frac{d\alpha_1\, d\alpha_2}{2\pi}\, e^{-(\alpha_1^2+\alpha_2^2)/2} e^{i(\beta_2\alpha_1-\beta_1\alpha_2)} = e^{-(\beta_1^2+\beta_2^2)/2} = e^{-|\beta|^2} \tag{103}$$

$$\int \frac{d^2\alpha}{\pi\sigma^2}\, e^{-|\alpha|^2/\sigma^2} D(\alpha,\beta) = \int \frac{d^2\alpha'}{\pi}\, e^{-|\alpha'|^2} D(\alpha',\sigma\beta) = e^{-\sigma^2|\beta|^2} \tag{104}$$

$$\begin{aligned}\int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} e^{\alpha\gamma^*-\alpha^*\beta} &= \int \frac{d\alpha_1\, d\alpha_2}{2\pi}\, e^{-(\alpha_1^2+\alpha_2^2)/2} e^{\alpha_1(\gamma^*-\beta)/\sqrt{2}} e^{i\alpha_2(\gamma^*+\beta)/\sqrt{2}} \\ &= e^{(\gamma^*-\beta)^2/4} e^{-(\gamma^*+\beta)^2/4} \\ &= e^{-\beta\gamma^*}\end{aligned} \tag{105}$$

$$\int \frac{d^2\alpha}{\pi\sigma^2}\, e^{-|\alpha|^2/\sigma^2} e^{\alpha\gamma^*-\alpha^*\beta} = \int \frac{d^2\alpha'}{\pi}\, e^{-|\alpha'|^2} e^{\alpha'\sigma\gamma^*-\alpha'^*\sigma\beta} = e^{-\sigma^2\beta\gamma^*} \tag{106}$$

11. Orthogonality and completeness of displacement operators

$$\operatorname{tr}\big(D(a,\alpha)\big) = \int \frac{d^2\beta}{\pi}\,\langle\beta|D(a,\alpha)|\beta\rangle = e^{-|\alpha|^2/2}\int \frac{d^2\beta}{\pi}\,D(\beta,\alpha) = \pi\delta^2(\alpha) \tag{107}$$

$$\operatorname{tr}\big(D^\dagger(a,\beta)D(a,\alpha)\big) = D(\beta,\alpha/2)\operatorname{tr}\big(D(a,\alpha-\beta)\big) = \pi\delta^2(\alpha-\beta) \tag{108}$$

$$\begin{aligned}\int \frac{d^2\alpha}{\pi}\,\langle\mu|D(a,\alpha)|\beta\rangle\langle\gamma|D^\dagger(a,\alpha)|\nu\rangle &= \langle\mu|\beta\rangle\langle\gamma|\nu\rangle\int\frac{d^2\alpha}{\pi}\,e^{-|\alpha|^2}e^{\alpha(\mu^*-\gamma^*)-\alpha^*(\beta-\nu)}\\ &= \langle\mu|\beta\rangle\langle\gamma|\nu\rangle e^{-(\beta-\nu)(\mu^*-\gamma^*)}\\ &= e^{-|\mu|^2/2}e^{-|\beta|^2/2}e^{\beta\mu^*}\\ &\qquad\times e^{-|\gamma|^2/2}e^{-|\nu|^2/2}e^{\nu\gamma^*}e^{-(\beta-\nu)(\mu^*-\gamma^*)}\\ &= e^{-|\beta|^2/2}e^{-|\gamma|^2/2}e^{\beta\gamma^*}e^{-|\nu|^2/2}e^{-|\mu|^2/2}e^{\nu\mu^*}\\ &= \langle\gamma|\beta\rangle\langle\mu|\nu\rangle\end{aligned} \tag{109}$$

$$\begin{aligned}\operatorname{tr}\left((a^\dagger)^k a^l\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\beta)\right) &= \int\frac{d^2\alpha}{\pi}\,(\alpha^*)^k\alpha^l D(\alpha,\beta)\\ &= \frac{\partial^{k+l}}{\partial\beta^k\partial(-\beta^*)^l}\underbrace{\int\frac{d^2\alpha}{\pi}\,D(\alpha,\beta)}_{=\pi\delta^2(\beta)=\operatorname{tr}(D(a,\beta))}\\ &= \frac{\partial^{k+l}}{\partial\beta^k\partial(-\beta^*)^l}\operatorname{tr}\big(e^{-\beta^* a}e^{\beta a^\dagger}\big)\\ &= \operatorname{tr}\big((a^\dagger)^k a^l e^{-\beta^* a}e^{\beta a^\dagger}\big)\end{aligned} \tag{110}$$

$$\begin{aligned}&\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\beta) = e^{-\beta^* a}e^{\beta a^\dagger} = e^{-|\beta|^2/2}D(a,\beta)\\ &\int\frac{d^2\beta}{\pi}\,e^{-|\beta|^2/2}D(a,\beta)D(\beta,\alpha) = |\alpha\rangle\langle\alpha|\end{aligned} \tag{111}$$

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}\,D(a,\alpha)|\beta\rangle\langle\gamma|D^\dagger(a,\alpha) &= \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)D(a,\beta)|0\rangle\langle 0|D^\dagger(a,\gamma)D^\dagger(a,\alpha)\\ &= D(a,\beta)\left(\int\frac{d^2\alpha}{\pi}\,D(a+\beta,\alpha)|0\rangle\langle 0|D^\dagger(a+\gamma,\alpha)\right)D^\dagger(a,\gamma)\\ &= D(a,\beta)\left(\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\gamma-\beta)\right)D^\dagger(a,\gamma)\\ &= e^{-|\gamma-\beta|^2/2}D(a,\beta)D(a,\gamma-\beta)D^\dagger(a,\gamma)\\ &= D(\gamma,\beta/2)e^{-|\beta-\gamma|^2/2}D(a,\beta)D^\dagger(a,\beta)D(a,\gamma)D^\dagger(a,\gamma)\\ &= \langle\gamma|\beta\rangle\,1\end{aligned} \tag{112}$$

$$\mathbf{I} = \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)\odot D^\dagger(a,\alpha) = \int\frac{d^2\alpha}{\pi}\,D^\dagger(a,\alpha)\odot D(a,\alpha) \tag{113}$$

[Follows from Eq. (109) or Eq. (112)]

$$\operatorname{tr}(A)1 = \mathbf{I}(A) = \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)AD^\dagger(a,\alpha) \tag{114}$$

$$1 = \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)\rho D^\dagger(a,\alpha) = \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)|\psi\rangle\langle\psi|D^\dagger(a,\alpha) \tag{115}$$

$$\mathrm{tr}(A) = \int \frac{d^2\alpha}{\pi}\, \mathrm{tr}\big(D(a,\alpha)\rho D^\dagger(a,\alpha) A\big) = \int \frac{d^2\alpha}{\pi}\, \langle\psi|D^\dagger(a,\alpha) A D(a,\alpha)|\psi\rangle \tag{116}$$

12. Operator ordering

$$D^{(s)}(a,\alpha) \equiv e^{s|\alpha|^2/2} D(a,\alpha) \tag{117}$$

$$D^{(s)\dagger}(a,\alpha) = D^{(s)}(-a,\alpha) = D^{(s)}(a,-\alpha) \tag{118}$$

$$\mathrm{tr}\big(D^{(s)}(a,\alpha)\big) = \pi\delta^2(\alpha) \tag{119}$$

$$\mathrm{tr}\big(D^{(-s)\dagger}(a,\beta) D^{(s)}(a,\alpha)\big) = \pi\delta^2(\alpha-\beta) \tag{120}$$

$$\mathbf{I} = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha) \odot D^{(s)}(a,\alpha) \tag{121}$$

$$\begin{aligned} s=+1 \text{ (normal ordering):} \quad & D^{(+1)}(a,\alpha) = e^{\alpha a^\dagger} e^{-\alpha^* a} = \boldsymbol{:} D(a,\alpha) \boldsymbol{:} \\ s=0 \text{ (symmetric ordering):} \quad & D^{(0)}(a,\alpha) = D(a,\alpha) = e^{\alpha a^\dagger - \alpha^* a} \\ s=-1 \text{ (antinormal ordering):} \quad & D^{(-1)}(a,\alpha) = e^{-\alpha^* a} e^{\alpha a^\dagger} \end{aligned} \tag{122}$$

$$\begin{aligned} [(a^\dagger)^k a^l]_{(s)} &\equiv \frac{\partial^{k+l} D^{(s)}(a,\alpha)}{\partial\alpha^k\, \partial(-\alpha^*)^l}\bigg|_{\alpha=0} \\ D^{(s)}(a,\alpha) &= \sum_{k,l} \frac{\alpha^k(-\alpha^*)^l}{k!\,l!} [(a^\dagger)^k a^l]_{(s)} \end{aligned} \tag{123}$$

$$\begin{aligned} [(a^\dagger)^k a^l]_{(+1)} &= (a^\dagger)^k a^l \\ [(a^\dagger)^k a^l]_{(-1)} &= a^l (a^\dagger)^k \end{aligned} \tag{124}$$

$$\begin{aligned} [(a^\dagger)^k a^k]_{(s)} &= \frac{\partial^{2k}[e^{(s-1)\alpha\alpha^*/2} D^{(+1)}(a,\alpha)]}{\partial\alpha^k\,\partial(-\alpha^*)^k}\bigg|_{\alpha=\alpha^*=0} \\ &= \sum_{m=0}^{k} \left(\frac{k!}{m!(k-m)!}\right)^2 \underbrace{\frac{\partial^{2m} e^{-(1-s)\alpha\alpha^*/2}}{\partial\alpha^m\,\partial(-\alpha^*)^m}\bigg|_{\alpha=\alpha^*=0}}_{= m!\, \frac{d^m e^{(1-s)x/2}}{dx^m}\big|_{x=0}} \underbrace{\frac{\partial^{2(k-m)} D^{(+1)}(a,\alpha)}{\partial\alpha^{k-m}\,\partial(-\alpha^*)^{k-m}}\bigg|_{\alpha=\alpha^*=0}}_{= (a^\dagger)^{k-m} a^{k-m}} \\ &= \sum_{m=0}^{k} \frac{1}{m!}\left(\frac{k!}{(k-m)!}\right)^2 \left(\frac{1-s}{2}\right)^m (a^\dagger)^{k-m} a^{k-m} \\ &= \sum_{m=0}^{k} \frac{1}{m!}\left(\frac{k!}{(k-m)!}\right)^2 \left(\frac{1-s}{2}\right)^m (a^\dagger a)^{(k-m)} \\ &= \left(\frac{1-s}{2}\right)^k k! \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \frac{(a^\dagger a)^{(m)}}{m!} \left(\frac{2}{1-s}\right)^m \end{aligned} \tag{125}$$

$$\begin{aligned} \tilde{D}^{(s)}(a,\beta) &\equiv \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a,\alpha) D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a-\beta,\alpha) \equiv \delta^{(s)2}(a-\beta) \\ D^{(s)}(a,\alpha) &= \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(s)}(a,\beta) D(\beta,\alpha) \end{aligned} \tag{126}$$

In general lingo, the operators $D^{(s)}(a,\alpha)$ are *frame operators*, and the Fourier transform is their expression in terms of *harmonic tensors*, which in this case are displacement operators.

$$\tilde{D}^{(s)\dagger}(a,\beta) = \tilde{D}^{(s)}(a,\beta) \tag{127}$$

$$\mathrm{tr}\big(\tilde{D}^{(s)}(a,\alpha)\big) = 1 \tag{128}$$

$$\int d^2\beta\,\tilde{D}^{(s)}(a,\beta) = D^{(s)}(a,0) = 1 \tag{129}$$

$$\begin{aligned}\mathrm{tr}\big(\tilde{D}^{(-s)\dagger}(a,\gamma)\tilde{D}^{(s)}(a,\beta)\big) &= \int \frac{d^2\alpha'}{\pi}\,\frac{d^2\alpha}{\pi}\,\underbrace{\mathrm{tr}\big(D^{(-s)\dagger}(a,\alpha')D^{(s)}(a,\alpha)\big)}_{=\pi\delta^2(\alpha-\alpha')}D(\alpha',-\gamma)D(\alpha,\beta)\\ &= \int \frac{d^2\alpha}{\pi}\,D(\alpha,\beta-\gamma)\\ &= \pi\delta^2(\beta-\gamma)\end{aligned} \tag{130}$$

$$\begin{aligned}\mathbf{I} &= \int \frac{d^2\alpha}{\pi}\,D^{(-s)\dagger}(a,\alpha)\odot D^{(s)}(a,\alpha) = \int \frac{d^2\alpha}{\pi}\,D^{(-s)}(a,\alpha)\odot D^{(s)\dagger}(a,\alpha)\\ &= \int \frac{d^2\alpha}{\pi}\,D^{(-s)\dagger}(a,\alpha)\odot \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)D(\beta,\alpha)\\ &= \int \frac{d^2\beta}{\pi}\,\left(\int \frac{d^2\alpha}{\pi}\,D^{(-s)}(a,\alpha)D(\alpha,\beta)\right)^{\dagger}\odot\tilde{D}^{(s)}(a,\beta)\\ &= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(-s)}(a,\beta)\odot\tilde{D}^{(s)}(a,\beta)\end{aligned} \tag{131}$$

$$\begin{aligned}[(a^\dagger)^k a^l]_{(s)} &= \left.\frac{\partial^{k+l}D^{(s)}(a,\alpha)}{\partial\alpha^k\,\partial(-\alpha^*)^l}\right|_{\alpha=0}\\ &= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)\,\underbrace{\left.\frac{\partial^{k+l}D(\beta,\alpha)}{\partial\alpha^k\,\partial(-\alpha^*)^l}\right|_{\alpha=0}}_{=(\beta^*)^k\beta^l}\\ &= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)(\beta^*)^k\beta^l\\ &= \Big([(a^\dagger)^l a^k]_{(s)}\Big)^{\dagger}\end{aligned} \tag{132}$$

$$\begin{aligned}\tilde{D}^{(s)}(a,\beta) &= \int \frac{d^2\alpha}{\pi}\,D^{(s)}(a,\alpha)D(\alpha,\beta)\\ &= \int \frac{d^2\alpha}{\pi}\,e^{(s-s')|\alpha|^2/2}D^{(s')}(a,\alpha)D(\alpha,\beta)\\ &= \int \frac{d^2\alpha}{\pi}\,e^{-(s'-s)|\alpha|^2/2}D(\alpha,\beta)\int \frac{d^2\gamma}{\pi}\,\tilde{D}^{(s')}(a,\gamma)D(\gamma,\alpha)\\ &= \int \frac{d^2\gamma}{\pi}\,\tilde{D}^{(s')}(a,\gamma)\underbrace{\int \frac{d^2\alpha}{\pi}\,e^{-(s'-s)|\alpha|^2/2}D(\alpha,\beta-\gamma)}_{=\frac{2}{s'-s}e^{-2|\beta-\gamma|^2/(s'-s)}}\\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\tilde{D}^{(s')}(a,\gamma)e^{-2|\gamma-\beta|^2/(s'-s)}\;,\quad s\le s'\end{aligned} \tag{133}$$

$$\mathrm{tr}\big(\tilde{D}^{(-s)}(a,\beta)[(a^\dagger)^k a^l]_{(s)}\big) = \int \frac{d^2\gamma}{\pi}\,\underbrace{\mathrm{tr}\big(\tilde{D}^{(-s)\dagger}(a,\beta)\tilde{D}^{(s)}(a,\gamma)\big)}_{=\pi\delta^2(\beta-\gamma)}(\gamma^*)^k\gamma^l = (\beta^*)^k\beta^l \tag{134}$$

$$\mathrm{tr}\big(\tilde{D}^{(-s)}(a,\beta)D^{(s)}(a,\alpha)\big) = \int \frac{d^2\gamma}{\pi}\, \underbrace{\mathrm{tr}\big(\tilde{D}^{(-s)\dagger}(a,\beta)\tilde{D}^{(s)}(a,\gamma)\big)}_{=\pi\delta^2(\beta-\gamma)} D(\gamma,\alpha) = D(\beta,\alpha) \tag{135}$$

$$\begin{aligned}\tilde{D}^{(s)}(a,\beta) &= \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a-\beta,\alpha)\\ &= D(a,\beta)\left(\int \frac{d^2\alpha}{\pi}\, D^{(s)}(a,\alpha)\right) D^\dagger(a,\beta)\\ &= D(a,\beta)\tilde{D}^{(s)}(a,0)D^\dagger(a,\beta)\end{aligned} \tag{136}$$

$$\begin{aligned}\tilde{D}^{(s)}(a,\beta+\gamma) &= D(a,\beta+\gamma)\tilde{D}^{(s)}(a,0)D^\dagger(a,\beta+\gamma)\\ &= D(a,\beta)D(a,\gamma)\tilde{D}^{(s)}(a,0)D^\dagger(a,\gamma)D^\dagger(a,\beta)\\ &= D(a,\beta)\tilde{D}^{(s)}(a,\gamma)D^\dagger(a,\beta)\end{aligned} \tag{137}$$

$$\begin{aligned}\tilde{D}^{(s)}(a,\beta) &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma)e^{-2|\gamma-\beta|^2/(s'-s)}\\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma+\beta)e^{-2|\gamma|^2/(s'-s)}\\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, D(a,\gamma)\tilde{D}^{(s')}(a,\beta)D^\dagger(a,\gamma)e^{-2|\gamma|^2/(s'-s)}\ , \quad s\le s'\end{aligned} \tag{138}$$

$$\begin{aligned}\langle\alpha|\tilde{D}^{(s)}(a,0)|\beta\rangle &= \int \frac{d^2\gamma}{\pi}\,\langle\alpha|D^{(s)}(a,\gamma)|\beta\rangle\\ &= \langle\alpha|\beta\rangle\int \frac{d^2\gamma}{\pi}\, e^{-(1-s)|\gamma|^2/2}e^{\gamma\alpha^*-\gamma^*\beta}\\ &= \langle\alpha|\beta\rangle\frac{2}{1-s}e^{-2\beta\alpha^*/(1-s)} \qquad s<1,\\ &= \frac{2}{1-s}\langle\alpha|e^{-\mu a^\dagger a}|\beta\rangle\ , \quad e^\mu = \frac{s-1}{s+1}\end{aligned} \tag{139}$$

$$\tilde{D}^{(s)}(a,0) = \frac{1}{Z}e^{-\mu a^\dagger a} = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^{a^\dagger a}\ , \quad Z = \frac{1-s}{2} = \frac{1}{1-e^{-\mu}}\ , \quad s = -\coth(\mu/2) \tag{140}$$

As s goes from $-\infty$ to -1_-, μ ranges over positive inverse temperatures, going from $\mu = 0_+$ at $s=-\infty$ to $\mu=\infty$ at $s=-1_-$. At $s=-1$, $\mu=\nu\pm i\pi$ acquires an imaginary part $\pm i\pi$; as s ranges from -1_+ to 1_-, ν ranges over the whole real line, going from $\nu=\infty$ at $s=-1_+$ to $\nu=0$ at $s=0$ and to $\nu=-\infty$ at $s=1_-$. At $s=+1$, μ becomes real again; as s ranges from 1_+ to ∞, μ ranges over negative inverse temperatures, going from $\mu=-\infty$ at $s=1_+$ to $\mu=0_-$ at $s=\infty$.

$$
\begin{aligned}
\tilde{D}^{(0)}(a,\gamma)\tilde{D}^{(0)}(a,\beta) &= \tilde{D}^{(0)\dagger}(a,\gamma)\tilde{D}^{(0)}(a,\beta)\\
&= \int \frac{d^2\alpha'}{\pi}\,\frac{d^2\alpha}{\pi}\,D(\alpha',-\gamma)D(\alpha,\beta)\,\underbrace{D^\dagger(a,\alpha')D(a,\alpha)}_{=D(\alpha',\alpha/2)D(a,\alpha-\alpha')}\\
&= \int \frac{d^2\mu}{\pi}\,D(a,\mu)\int\frac{d^2\nu}{\pi}\,D(\nu,\mu/2)D(\nu-\mu/2,-\gamma)D(\nu+\mu/2,\beta)\\
&= \int \frac{d^2\mu}{\pi}\,D(a,\mu)D\big(\mu,(\beta+\gamma)/2\big)\underbrace{\int\frac{d^2\nu}{\pi}\,D(\nu,\mu/2+\beta-\gamma)}_{=4\pi\delta^2\big(\mu-2(\gamma-\beta)\big)}\\
&= 4D\big(a,2(\gamma-\beta)\big)D(\gamma-\beta,\beta+\gamma)\\
&= 4D\big(a,2(\gamma-\beta)\big)D(\gamma,2\beta)\\
&= 4D^\dagger(a,-2\gamma)D(a,-2\beta)
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
[\tilde{D}^{(0)}(a,\alpha),D(a,\beta)] &= \int\frac{d^2\gamma}{\pi}D(\gamma,\alpha)[D(a,\gamma),D(a,\beta)]\\
&= \int\frac{d^2\gamma}{\pi}\big(D(\gamma,\alpha-\beta/2)-D(\gamma,\alpha+\beta/2)\big)D(a,\gamma+\beta)\\
&= D(\alpha,\beta)\int\frac{d^2\gamma}{\pi}D(a,\gamma)\big(D(\gamma,\alpha-\beta/2)-D(\gamma,\alpha+\beta/2)\big)\\
&= D(\alpha,\beta)\big(\tilde{D}^{(0)}(a,\alpha-\beta/2)-\tilde{D}^{(0)}(a,\alpha+\beta/2)\big)
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
[\tilde{D}^{(0)}(a,\alpha),\tilde{D}^{(0)}(a,\beta)] &= \int\frac{d^2\gamma}{\pi}D(\gamma,\beta)[\tilde{D}^{(0)}(a,\alpha),D(a,\gamma)]\\
&= \int\frac{d^2\gamma}{\pi}D(\gamma,\beta)D(\alpha,\gamma)\big(\tilde{D}^{(0)}(a,\alpha-\gamma/2)-\tilde{D}^{(0)}(a,\alpha+\gamma/2)\big)\\
&= \int\frac{d^2\gamma}{\pi}\tilde{D}^{(0)}(a,\alpha+\gamma/2)\big(D(\gamma,\alpha-\beta)-D(\gamma,\beta-\alpha)\big)\\
&= 4\int\frac{d^2\gamma}{\pi}\tilde{D}^{(0)}(a,\gamma)\\
&\qquad\times\Big(D\big(\gamma,2(\alpha-\beta)\big)D(\alpha,2\beta)-D\big(\gamma,2(\beta-\alpha)\big)D(\alpha,-2\beta)\Big)\\
&= 4\Big[D\big(a,2(\alpha-\beta)\big)D(\alpha,2\beta)-D\big(a,2(\beta-\alpha)\big)D(\alpha,-2\beta)\Big]
\end{aligned}
\tag{143}
$$

$$
\begin{aligned}
[\tilde{D}^{(0)}(a,\alpha),\tilde{D}^{(0)}(a,\beta)] &= 4D\big(a,2(\alpha-\beta)\big)D(\alpha,2\beta)-4D\big(a,2(\beta-\alpha)\big)D(\beta,2\alpha)\\
&= 4\Big[D\big(a,2(\alpha-\beta)\big)D(\alpha,2\beta)-D\big(a,2(\beta-\alpha)\big)D(\alpha,-2\beta)\Big]
\end{aligned}
\tag{144}
$$

$$
s=0\text{:}\;\left\{
\begin{aligned}
&\langle\alpha|\tilde{D}^{(0)}(a,0)|\beta\rangle=\langle\alpha|\beta\rangle 2e^{-2\beta\alpha^*}=2\langle\alpha|-\beta\rangle\quad\Longrightarrow\quad \tilde{D}^{(0)}(a,0)=2P\\
&\tilde{D}^{(0)}(a,\beta)=2D(a,\beta)PD^\dagger(a,\beta)=2PD^\dagger(a,2\beta)=2PD(a,-2\beta)\\
&\langle\alpha|\tilde{D}^{(0)}(a,\gamma)|\beta\rangle=2\langle-\alpha|D(a,-2\gamma)|\beta\rangle=2D(\gamma,\beta-\alpha)D(\beta,\alpha/2)e^{-2|\gamma-(\alpha+\beta)/2|^2}\\
&\tilde{D}^{(0)}(a,\beta)=2\int\frac{d^2\alpha}{\pi}D(a,\beta)|\alpha\rangle\langle-\alpha|D^\dagger(a,\beta)=2\int\frac{d^2\alpha}{\pi}|\beta+\alpha\rangle\langle\beta-\alpha|D(\alpha,\beta)
\end{aligned}\right.
\tag{145}
$$

$$
\begin{aligned}
\langle x|\tilde{D}^{(0)}(a,\beta)|x'\rangle=\langle x|2PD(a,-2\beta)|x'\rangle &= 2\langle -x|D(a,-2\beta)|x'\rangle\\
&= e^{-2i\beta_1\beta_2}e^{2i\beta_2 x}\delta\big(\beta_1-(x+x')/2\big)\\
&= e^{i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big)
\end{aligned}
\tag{146}
$$

$$\begin{aligned}\langle p|\tilde{D}^{(0)}(a,\beta)|p'\rangle = \langle p|2PD(a,-2\beta)|p'\rangle &= 2\langle -p|D(a,-2\beta)|p'\rangle \\ &= e^{2i\beta_1\beta_2}e^{-2i\beta_1 p}\delta\big(\beta_2-(p+p')/2\big) \\ &= e^{-i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big)\end{aligned} \tag{147}$$

$$s=-1\text{:}\quad\left\{\begin{array}{l}\langle\alpha|\tilde{D}^{(-1)}(a,0)|\beta\rangle = \langle\alpha|\beta\rangle e^{-\beta\alpha^*} = e^{-|\alpha|^2/2}e^{-|\beta|^2/2} = \langle\alpha|0\rangle\langle 0|\beta\rangle \\ \quad\Longrightarrow\quad \tilde{D}^{(-1)}(a,0) = |0\rangle\langle 0| \\ \tilde{D}^{(-1)}(a,\beta) = D(a,\beta)|0\rangle\langle 0|D^\dagger(a,\beta) = |\beta\rangle\langle\beta| \\ |\beta\rangle\langle\beta| = \displaystyle\int\frac{d^2\alpha}{\pi}\,e^{-\alpha^* a}e^{\alpha a^\dagger}D(\alpha,\beta) \qquad e^{-\alpha^* a}e^{\alpha a^\dagger} = \int\frac{d^2\beta}{\pi}\,|\beta\rangle\langle\beta|D(\beta,\alpha)\end{array}\right. \tag{148}$$

$$\begin{aligned}\mathcal{G}^{(s)} &= \int\frac{d^2\alpha}{\pi}\,D^{(s)\dagger}(a,\alpha)\odot D^{(s)}(a,\alpha) = \int\frac{d^2\alpha}{\pi}\,D^{(s)}(a,\alpha)\odot D^{(s)\dagger}(a,\alpha) \\ &= \int\frac{d^2\alpha}{\pi}\,e^{s|\alpha|^2}D^\dagger(a,\alpha)\odot D(a,\alpha) = \int\frac{d^2\alpha}{\pi}\,e^{s|\alpha|^2}D(a,\alpha)\odot D^\dagger(a,\alpha) \\ &= \int\frac{d^2\alpha}{\pi}\,D^{(s)\dagger}(a,\alpha)\odot\int\frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)D(\beta,\alpha) \\ &= \int\frac{d^2\beta}{\pi}\left(\int\frac{d^2\alpha}{\pi}\,D^{(s)}(a,\alpha)D(\alpha,\beta)\right)^\dagger\odot\tilde{D}^{(s)}(a,\beta) \\ &= \int\frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)\odot\tilde{D}^{(s)}(a,\beta)\end{aligned} \tag{149}$$

$\mathcal{G}^{(s)}$ is the *quartic twirl* of the *frame operators* $\tilde{D}^{(s)}(a,\beta)$.

$$\begin{aligned}\mathcal{G}^{(0)} &= \mathbf{I} \\ \mathcal{G}^{(-1)} &= \int\frac{d^2\beta}{\pi}\,|\beta\rangle\langle\beta|\odot|\beta\rangle\langle\beta|\end{aligned} \tag{150}$$

$$\begin{aligned}\mathcal{G}^{(s)}\big(D(a,\alpha)\big) &= \int\frac{d^2\gamma}{\pi}\,e^{s|\gamma|^2}D^\dagger(a,\gamma)D(a,\alpha)D(a,\gamma) \\ &= D(a,\alpha)\int\frac{d^2\gamma}{\pi}\,e^{s|\gamma|^2}D(\gamma,\alpha) \\ &= D(a,\alpha)\left(-\frac{1}{s}e^{|\alpha|^2/s}\right),\quad s\le 0\end{aligned} \tag{151}$$

$$\lim_{s\to-\infty}\big[-s\mathcal{G}^{(s)}\big(D(a,\alpha)\big)\big] = D(a,\alpha) \qquad\Longrightarrow\qquad \lim_{s\to-\infty}\big(-s\mathcal{G}^{(s)}\big) = \mathcal{I} \tag{152}$$

$$1 = -s\mathcal{G}^{(s)}(1) = -s\int\frac{d^2\beta}{\pi}\,\big(\tilde{D}^{(s)}(a,\beta)\big)^2,\quad s\le 0\,. \tag{153}$$

$$\mathcal{G}^{(s)}\big|D(a,\alpha)\big) = \int\frac{d^2\beta}{\pi}\,D^{(s)}(a,\beta)\mathrm{tr}\big(D^{(s)\dagger}(a,\beta)D(a,\alpha)\big) = e^{s|\alpha|^2}D(a,\alpha) \tag{154}$$

$$1 = \mathcal{G}^{(s)}|1) = \int\frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta) \tag{155}$$

13. Operators and associated functions

$$A = \mathbf{I}|A) = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha)\, \underbrace{\mathrm{tr}\big(AD^{(s)}(a,\alpha)\big)}_{\equiv F_A^{(s)}(\alpha)} = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha) F_A^{(s)}(\alpha) \tag{156}$$

$$F_A^{(s)}(\alpha) = F_{A^\dagger}^{(s)*}(-\alpha) = e^{s|\alpha|^2/2} F_A^{(0)}(\alpha) \tag{157}$$

$$F_A^{(s)}(0) = \mathrm{tr}(A) \tag{158}$$

$$F_P^{(s)}(\alpha) = e^{s|\alpha|^2/2} F_P^{(0)}(\alpha) = \frac{1}{2} e^{s|\alpha|^2/2} \tag{159}$$

$$\begin{aligned} \mathrm{tr}(A^\dagger B) &= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\, \underbrace{\mathrm{tr}\big(D^{(s)}(a,\alpha) D^{(-s)\dagger}(a,\beta)\big)}_{=\pi\delta^2(\beta-\alpha)} F_A^{(-s)*}(\alpha) F_B^{(s)}(\beta) \\ &= \int \frac{d^2\alpha}{\pi}\, F_A^{(-s)*}(\alpha) F_B^{(s)}(\alpha) \end{aligned} \tag{160}$$

$$\begin{aligned} A^\dagger B &= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\, D(a,\alpha) D^\dagger(a,\beta) F_A^{(0)*}(\alpha) F_B^{(0)}(\beta) \\ &= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\, D(\beta,-\alpha/2) D^\dagger(a,\beta-\alpha) F_A^{(0)*}(\alpha) F_B^{(0)}(\beta) \\ &= \int \frac{d^2\mu}{\pi}\, D^\dagger(a,\mu) \underbrace{\int \frac{d^2\nu}{\pi}\, D(\nu,\mu/2) F_A^{(0)*}(\nu-\mu/2) F_B^{(0)}(\nu+\mu/2)}_{=F_{A^\dagger B}^{(0)}(\mu)} \end{aligned} \tag{161}$$

$$A = \mathbf{I}|A) = \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(-s)}(a,\beta)\, \underbrace{\mathrm{tr}\big(A\tilde{D}^{(s)}(a,\beta)\big)}_{\equiv \tilde{F}_A^{(s)}(\beta)} = \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(-s)}(a,\beta) \tilde{F}_A^{(s)}(\beta) \tag{162}$$

$$\tilde{F}_A^{(s)}(\beta) = \tilde{F}_{A^\dagger}^{(s)*}(\beta) \tag{163}$$

$$\mathrm{tr}(A) = \int \frac{d^2\beta}{\pi}\, \tilde{F}_A^{(s)}(\beta) \tag{164}$$

$$\mathrm{tr}(A^\dagger B) = \int \frac{d^2\beta}{\pi}\,\frac{d^2\gamma}{\pi}\, \underbrace{\mathrm{tr}\big(\tilde{D}^{(s)}(a,\beta)\tilde{D}^{(-s)}(a,\gamma)\big)}_{=\pi\delta^2(\gamma-\beta)} \tilde{F}_A^{(-s)*}(\beta)\tilde{F}_B^{(s)}(\gamma) = \int \frac{d^2\beta}{\pi}\, \tilde{F}_A^{(-s)*}(\beta)\tilde{F}_B^{(s)}(\beta) \tag{165}$$

$$\begin{aligned} A^\dagger B &= \int \frac{d^2\beta}{\pi}\,\frac{d^2\gamma}{\pi}\, \tilde{F}_A^{(0)*}(\beta)\tilde{F}_B^{(0)}(\gamma)\, \underbrace{\tilde{D}^{(0)}(a,\beta)\tilde{D}^{(0)}(a,\gamma)}_{=4D^\dagger\left(a,2(\gamma-\beta)\right)D(\beta,2\gamma)} \\ &= \int \frac{d^2\mu}{\pi}\, D^\dagger(a,\mu) \underbrace{\int \frac{d^2\nu}{\pi}\, D(\nu,\mu)\tilde{F}_A^{(0)*}(\nu-\mu/4)\tilde{F}_B^{(0)}(\nu+\mu/4)}_{=F_{A^\dagger B}^{(0)}(\mu)} \end{aligned} \tag{166}$$

$$\tilde{F}_A^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi}\, F_A^{(s)}(\alpha) D(\alpha,\beta) \qquad F_A^{(s)}(\alpha) = \int \frac{d^2\beta}{\pi}\, \tilde{F}_A^{(s)}(\beta) D(\beta,\alpha) \tag{167}$$

$$\begin{aligned}\tilde F_A^{(s)}(\beta) &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\tilde F_A^{(s')}(\gamma)e^{-2|\gamma-\beta|^2/(s'-s)}\\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\tilde F_A^{(s')}(\beta+\gamma)e^{-2|\gamma|^2/(s'-s)}\ , \quad s\le s'\end{aligned} \tag{168}$$

$$\begin{aligned}F_{A^\dagger B}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi}\,D(\nu,\mu/2)F_A^{(0)*}(\nu-\mu/2)F_B^{(0)}(\nu+\mu/2)\\ &= \int \frac{d^2\nu}{\pi}\,D(\nu,\mu)\tilde F_A^{(0)*}(\nu-\mu/4)\tilde F_B^{(0)}(\nu+\mu/4)\end{aligned} \tag{169}$$

$$\begin{aligned}F_{-i[A,B]}^{(0)}(\mu) &= -i\int \frac{d^2\nu}{\pi}\,[D(\nu,\mu/2)-D^*(\nu,\mu/2)]F_A^{(0)}(\mu/2-\nu)F_B^{(0)}(\mu/2+\nu)\\ &= 2\int \frac{d^2\nu}{\pi}\,\sin\!\bigg(i\,\frac{\nu\mu^*-\nu^*\mu}{2}\bigg)\,F_A^{(0)}(\mu/2-\nu)F_B^{(0)}(\mu/2+\nu)\end{aligned} \tag{170}$$

$$\begin{aligned}\tilde F_{A^\dagger B}^{(0)}(\alpha) &= \int \frac{d^2\mu}{\pi} F_{A^\dagger B}^{(0)}(\mu)D(\mu,\alpha)\\ &= \int \frac{d^2\mu}{\pi}\,\frac{d^2\nu}{\pi}\,D(\mu,\alpha-\nu)\tilde F_A^{(0)*}(\nu-\mu/4)\tilde F_B^{(0)}(\nu+\mu/4)\\ &= 4\int \frac{d^2\gamma}{\pi}\,\frac{d^2\delta}{\pi}\,D(\gamma,2\delta)\tilde F_A^{(0)*}(\alpha+\gamma)\tilde F_B^{(0)}(\alpha+\delta)\end{aligned} \tag{171}$$

$$\begin{aligned}\tilde F_{-i[A,B]}^{(0)}(\alpha) &= -4i\int \frac{d^2\gamma}{\pi}\,\frac{d^2\delta}{\pi}\,[D(\gamma,2\delta)-D^*(\gamma,2\delta)]\tilde F_A^{(0)}(\alpha+\gamma)\tilde F_B^{(0)}(\alpha+\delta)\\ &= 8\int \frac{d^2\gamma}{\pi}\,\frac{d^2\delta}{\pi}\,\sin[2i(\gamma\delta^*-\gamma^*\delta)]\tilde F_A^{(0)}(\alpha+\gamma)\tilde F_B^{(0)}(\alpha+\delta)\\ &= \frac{1}{2}\int \frac{d^2\mu}{\pi}\,\frac{d^2\nu}{\pi}\,\sin\!\bigg(i\frac{\mu\nu^*-\mu^*\nu}{2}\bigg)\,\tilde F_A^{(0)}(\alpha+\mu/2)\tilde F_B^{(0)}(\alpha+\nu/2)\end{aligned} \tag{172}$$

$$\begin{aligned}\tilde F_A^{(s)}(\beta) &= \operatorname{tr}\big(A\tilde D^{(s)}(a,\beta)\big)\\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\operatorname{tr}\big(AD(a,\gamma)\tilde D^{(s')}(a,\beta)D^\dagger(a,\gamma)\big)e^{-2|\gamma|^2/(s'-s)}\\ &= \operatorname{tr}\Bigg(\underbrace{\bigg(\frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,D^\dagger(a,\gamma)AD(a,\gamma)e^{-2|\gamma|^2/(s'-s)}\bigg)}_{=A'}\tilde D^{(s')}(a,\beta)\Bigg)\\ &= \tilde F_{A'}^{(s')}(\beta)\ , \quad s\le s'\end{aligned} \tag{173}$$

$$A=\sum_{k,l} f_{kl}^{(-s)}[(a^\dagger)^k a^l]_{(-s)} \quad\Longleftrightarrow\quad \tilde F_A^{(s)}(\beta)=\sum_{k,l} f_{kl}^{(-s)}(\beta^*)^k\beta^l \tag{174}$$

$$s=+1\colon\ \begin{cases} A=\displaystyle\int \frac{d^2\beta}{\pi}\,\tilde D^{(-1)}(a,\beta)\tilde F_A^{(+1)}(\beta)=\int \frac{d^2\beta}{\pi}\,|\beta\rangle\langle\beta|\tilde F_A^{(+1)}(\beta)\\ \tilde F_A^{(+1)}(\beta)=\displaystyle\int \frac{d^2\alpha}{\pi}\,F_A^{(+1)}(\alpha)D(\alpha,\beta)=\int \frac{d^2\alpha}{\pi}\,\operatorname{tr}\big(Ae^{\alpha a^\dagger}e^{-\alpha^* a}\big)D(\alpha,\beta)\end{cases} \tag{175}$$

$$s=0\colon\ \begin{cases} A=\displaystyle\int \frac{d^2\beta}{\pi}\,\tilde D^{(0)}(a,\beta)\tilde F_A^{(0)}(\beta)=2P\int \frac{d^2\beta}{\pi}\,D(a,-2\beta)\tilde F_A^{(0)}(\beta)\\ \tilde F_A^{(0)}(\beta)=\displaystyle\int \frac{d^2\alpha}{\pi}\,F_A^{(0)}(\alpha)D(\alpha,\beta)=\int \frac{d^2\alpha}{\pi}\,\operatorname{tr}\big(AD(a,\alpha)\big)D(\alpha,\beta)\\ \tilde F_A^{(0)}(\beta)=\operatorname{tr}\big(A\tilde D^{(0)}(a,\beta)\big)=2\operatorname{tr}\big(APD(a,-2\beta)\big)=2\displaystyle\int \frac{d^2\alpha}{\pi}\langle\beta-\alpha|A|\beta+\alpha\rangle D(\alpha,\beta)\end{cases} \tag{176}$$

$$\tilde{F}_P^{(0)}(\beta) = 2\,\mathrm{tr}\big(D(a,-2\beta)\big) = 2\pi\delta^2(-2\beta) = \frac{\pi}{2}\delta^2(\beta) \tag{177}$$

$$\begin{aligned}
\langle x|A|x'\rangle &= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,\langle x|2PD(a,-2\beta)|x'\rangle \\
&= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,e^{i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big) \\
&= \int \frac{d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}\Big((x+x')/2+i\beta_2\Big)\bigg)\,e^{i\beta_2(x-x')}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_A^{(0)}(\beta) &= 2\int dx\,\langle x|APD(a,-2\beta)|x\rangle \\
&= \int dx\,dx'\,\langle x|A|x'\rangle\langle x'|2PD(a,-2\beta)|x\rangle \\
&= \int dx\,dx'\,\langle x|A|x'\rangle e^{-i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big) \\
&= \int dX\,d\xi\,\langle X+\xi/2|A|X-\xi/2\rangle e^{-i\beta_2\xi}\delta\big(\beta_1-X\big) \\
&= \int d\xi\,\langle \beta_1+\xi/2|A|\beta_1-\xi/2\rangle e^{-i\beta_2\xi}
\end{aligned} \tag{178}$$

$$\begin{aligned}
\langle p|A|p'\rangle &= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,\langle p|2PD(a,-2\beta)|p'\rangle \\
&= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,e^{-i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big) \\
&= \int \frac{d\beta_1}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}\Big(\beta_1+i(p+p')/2\Big)\bigg)\,e^{-i\beta_1(p-p')}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_A^{(0)}(\beta) &= 2\int dp\,\langle p|APD(a,-2\beta)|p\rangle \\
&= \int dp\,dp'\,\langle p|A|p'\rangle\langle p'|2PD(a,-2\beta)|p\rangle \\
&= \int dp\,dp'\,\langle p|A|p'\rangle e^{i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big) \\
&= \int dP\,d\eta\,\langle P+\eta/2|A|P-\eta/2\rangle e^{i\beta_1\eta}\delta\big(\beta_2-P\big) \\
&= \int d\eta\,\langle \beta_2+\eta/2|A|\beta_2-\eta/2\rangle e^{i\beta_1\eta}
\end{aligned} \tag{179}$$

$$s=-1\colon\ \left\{\begin{aligned}
&A = \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(+1)}(a,\beta)\tilde{F}_A^{(-1)}(\beta) \\
&\tilde{F}_A^{(-1)}(\beta) = \mathrm{tr}\big(A\tilde{D}^{(-1)}(a,\beta)\big) = \langle\beta|A|\beta\rangle
\end{aligned}\right. \tag{180}$$

$$\tilde{F}_P^{(-1)}(\beta) = \langle\beta|P|\beta\rangle = \langle\beta|{-\beta}\rangle = e^{-2|\beta|^2} \tag{181}$$

$$\begin{aligned}
F^{(s)}_{aA}(\alpha) &= e^{(s-1)|\alpha|^2/2} F^{(+1)}_{aA}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial(-\alpha^*)} F^{(+1)}_{A}(\alpha) \\
&= -e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha^*} e^{-(s-1)|\alpha|^2/2} F^{(s)}_{A}(\alpha) \\
&= \left(-\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha\right) F^{(s)}_{A}(\alpha) \\
F^{(s)}_{a^\dagger A}(\alpha) &= e^{(s+1)|\alpha|^2/2} F^{(-1)}_{a^\dagger A}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} F^{(-1)}_{A}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} e^{-(s+1)|\alpha|^2/2} F^{(s)}_{A}(\alpha) \\
&= \left(\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^*\right) F^{(s)}_{A}(\alpha) \\
F^{(s)}_{Aa^\dagger}(\alpha) &= e^{(s-1)|\alpha|^2/2} F^{(+1)}_{Aa^\dagger}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} F^{(+1)}_{A}(\alpha) \\
&= e^{(s-1)|\alpha|^2/2} \frac{\partial}{\partial\alpha} e^{-(s-1)|\alpha|^2/2} F^{(s)}_{A}(\alpha) \\
&= \left(\frac{\partial}{\partial\alpha} + \frac{1-s}{2}\alpha^*\right) F^{(s)}_{A}(\alpha) \\
F^{(s)}_{Aa}(\alpha) &= e^{(s+1)|\alpha|^2/2} F^{(-1)}_{Aa}(\alpha) \\
&= e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial(-\alpha^*)} F^{(-1)}_{A}(\alpha) \\
&= -e^{(s+1)|\alpha|^2/2} \frac{\partial}{\partial\alpha^*} e^{-(s+1)|\alpha|^2/2} F^{(s)}_{A}(\alpha) \\
&= \left(-\frac{\partial}{\partial\alpha^*} + \frac{1+s}{2}\alpha\right) F^{(s)}_{A}(\alpha)
\end{aligned} \tag{182}$$

$$\begin{aligned}
\tilde{F}^{(s)}_{aA}(\beta) &= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{aA}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, \left[\left(-\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha\right) F^{(s)}_{A}(\alpha)\right] D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{A}(\alpha) \left(\frac{\partial}{\partial\alpha^*} - \frac{1-s}{2}\alpha\right) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{A}(\alpha) \left(\beta + \frac{1-s}{2}\frac{\partial}{\partial\beta^*}\right) D(\alpha,\beta) \\
&= \left(\beta + \frac{1-s}{2}\frac{\partial}{\partial\beta^*}\right) \tilde{F}^{(s)}_{A}(\beta) \\
\tilde{F}^{(s)}_{a^\dagger A}(\beta) &= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{a^\dagger A}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, \left[\left(\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^*\right) F^{(s)}_{A}(\alpha)\right] D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{A}(\alpha) \left(-\frac{\partial}{\partial\alpha} - \frac{1+s}{2}\alpha^*\right) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, F^{(s)}_{A}(\alpha) \left(\beta^* - \frac{1+s}{2}\frac{\partial}{\partial\beta}\right) D(\alpha,\beta) \\
&= \left(\beta^* - \frac{1+s}{2}\frac{\partial}{\partial\beta}\right) \tilde{F}^{(s)}_{A}(\beta) \\
\tilde{F}^{(s)}_{Aa^\dagger}(\beta) &= \tilde{F}^{(s)*}_{aA^\dagger}(\beta) = \left(\beta^* + \frac{1-s}{2}\frac{\partial}{\partial\beta}\right) \tilde{F}^{(s)}_{A}(\beta) \\
\tilde{F}^{(s)}_{Aa}(\beta) &= \tilde{F}^{(s)*}_{a^\dagger A^\dagger}(\beta) = \left(\beta - \frac{1+s}{2}\frac{\partial}{\partial\beta^*}\right) \tilde{F}^{(s)}_{A}(\beta)
\end{aligned} \tag{183}$$

$$\begin{aligned}
\mathcal{G}^{(0)} &= \mathbf{I} \\
\mathcal{G}^{(-1)} &= \int \frac{d^2\beta}{\pi}\, |\beta\rangle\langle\beta| \odot |\beta\rangle\langle\beta|
\end{aligned} \tag{184}$$

$$\begin{aligned}
\mathcal{G}^{(s)}|A) &= \int \frac{d^2\alpha}{\pi}\, D^{(s)\dagger}(a,\alpha)\mathrm{tr}\big(D^{(s)}(a,\alpha)A\big) = \int \frac{d^2\alpha}{\pi}\, D^{(s)\dagger}(a,\alpha) F^{(s)}_{A}(\alpha) \\
&= \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(s)}(a,\beta)\mathrm{tr}\big(\tilde{D}^{(s)}(a,\beta)A\big) = \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(s)}(a,\beta)\tilde{F}^{(s)}_{A}(\beta)
\end{aligned} \tag{185}$$

$$\mathcal{G}^{(s)}\big(D^{(s')}(a,\alpha)\big) = D^{(s')}(a,\alpha)\left(-\frac{1}{s}e^{|\alpha|^2/s}\right) = -\frac{1}{s}D(a,\alpha)e^{|\alpha|^2(1/s+s'/2)} = -\frac{1}{s}D^{(s'+2/s)}(a,\alpha)\,, \quad s \le 0 \tag{186}$$

$$\begin{aligned}
\mathcal{G}^{(s)}(A) &= -\frac{1}{s}\int \frac{d^2\alpha}{\pi}\, D^{(2/s-s')\dagger}(a,\alpha) F^{(s')}_{A}(\alpha) \\
&= -\frac{1}{s}\int \frac{d^2\beta}{\pi}\, \tilde{D}^{(2/s-s')}(a,\beta)\tilde{F}^{(s')}_{A}(\beta) \\
&= -\frac{1}{s}\int \frac{d^2\beta}{\pi}\, D(a,\beta)\tilde{D}^{(2/s-s')}(a,0)D^\dagger(a,\beta)\tilde{F}^{(s')}_{A}(\beta) \\
&= \int \frac{d^2\beta}{\pi}\, D(a,\beta)(s+1)^{a^\dagger a}D^\dagger(a,\beta)\tilde{F}^{(-1)}_{A}(\beta)
\end{aligned} \tag{187}$$

$$\mathcal{G}^{(s)}\big(\tilde{D}^{(s')}(a,\beta)\big) = -\frac{1}{s}\tilde{D}^{(s'+2/s)}(a,\beta) \tag{188}$$

$$\mathcal{G}^{(s)}\left(\left(\frac{s'+1}{s'-1}\right)^{a^\dagger a}\right) = -\frac{1}{s}\frac{1-s'}{1-s'-2/s}\left(\frac{s'+2/s+1}{s'+2/s-1}\right)^{a^\dagger a} \tag{189}$$

14. Characteristic functions and quasiprobability distributions

$$\begin{pmatrix} s\text{-ordered} \\ \text{characteristic} \\ \text{function} \end{pmatrix} = \Phi_\rho^{(s)}(\alpha) \equiv F_\rho^{(s)}(\alpha) = \mathrm{tr}\big(\rho D^{(s)}(a,\alpha)\big) = \Phi_\rho^{(s)*}(-\alpha) = e^{s|\alpha|^2/2}\Phi_\rho^{(0)}(\alpha) \tag{190}$$

$$\rho = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha)\Phi_\rho^{(s)}(\alpha) \tag{191}$$

$$\Phi_\rho^{(s)}(\alpha) = \sum_{k,l} \frac{\alpha^k(-\alpha^*)^l}{k!\,l!}\mathrm{tr}\Big(\rho[(a^\dagger)^k a^l]_{(s)}\Big) \tag{192}$$

$$\mathrm{tr}\Big(\rho[(a^\dagger)^k a^l]_{(s)}\Big) = \left.\frac{\partial^{k+l}\Phi_\rho^{(s)}(\alpha)}{\partial\alpha^k\,\partial(-\alpha^*)^l}\right|_{\alpha=0} \tag{193}$$

$$\Phi_\rho^{(s)}(0) = \mathrm{tr}(\rho) = 1 \tag{194}$$

$$\mathrm{tr}(\rho_1\rho_2) = \int \frac{d^2\alpha}{\pi}\, \Phi_{\rho_1}^{(-s)*}(\alpha)\Phi_{\rho_2}^{(s)}(\alpha) \tag{195}$$

$$\mathrm{tr}(\rho^2) = \int \frac{d^2\alpha}{\pi}\, \Phi_\rho^{(-s)*}(\alpha)\Phi_\rho^{(s)}(\alpha) \tag{196}$$

$$|\Phi_\rho^{(s)}(\alpha)| \le e^{s|\alpha|^2/2} \tag{197}$$

$$\Phi_\rho^{(s)}(\alpha) = e^{(s-s')|\alpha|^2/2}\Phi_\rho^{(s')}(\alpha) \tag{198}$$

$$\begin{pmatrix} s\text{-ordered} \\ \text{quasiprobability} \\ \text{distribution} \end{pmatrix} = W_\rho^{(s)}(\beta) \equiv \frac{1}{\pi}\tilde F_\rho^{(s)}(\beta) = \frac{1}{\pi}\mathrm{tr}\big(\rho\tilde D^{(s)}(a,\beta)\big) = W_\rho^{(s)*}(\beta) \tag{199}$$

$$\rho = \int d^2\beta\, \tilde D^{(-s)}(a,\beta)W_\rho^{(s)}(\beta) \tag{200}$$

$$\mathrm{tr}\Big(\rho[(a^\dagger)^k a^l]_{(s)}\Big) = \int d^2\beta\,\mathrm{tr}\Big(\tilde D^{(-s)}(a,\beta)[(a^\dagger)^k a^l]_{(s)}\Big)W_\rho^{(s)}(\beta) = \int d^2\beta\,(\beta^*)^k\beta^l W_\rho^{(s)}(\beta) \tag{201}$$

$$\mathrm{tr}(\rho A) = \int \frac{d^2\beta}{\pi}\, \tilde F_\rho^{(-s)*}(\beta)\tilde F_A^{(s)}(\beta) = \int d^2\beta\, \tilde F_A^{(s)}(\beta)W_\rho^{(-s)}(\beta) \tag{202}$$

$$1 = \mathrm{tr}(\rho) = \int d^2\beta\, W_\rho^{(s)}(\beta) \tag{203}$$

$$\mathrm{tr}(\rho_1\rho_2) = \pi\int d^2\beta\, W_{\rho_1}^{(-s)}(\beta)W_{\rho_2}^{(s)}(\beta) \tag{204}$$

$$\mathrm{tr}(\rho^2) = \pi\int d^2\beta\, W_\rho^{(-s)}(\beta)W_\rho^{(s)}(\beta) \tag{205}$$

$$W_\rho^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(s)}(\alpha)D(\alpha,\beta) \qquad \Phi_\rho^{(s)}(\alpha) = \int d^2\beta\, W_\rho^{(s)}(\beta)D(\beta,\alpha) \tag{206}$$

$$\begin{aligned} W_\rho^{(s)}(\beta) &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, W_\rho^{(s')}(\gamma)e^{-2|\gamma-\beta|^2/(s'-s)} \\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, W_\rho^{(s')}(\beta+\gamma)e^{-2|\gamma|^2/(s'-s)}\,, \quad s\le s' \end{aligned} \tag{207}$$

$$
\begin{aligned}
&W_\rho^{(s)}(\beta) = W_{\rho'}^{(s')}(\beta)\ ,\ \text{for } s \le s', \text{ where} \\
&\qquad \rho' = \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\, D^\dagger(a,\gamma)\rho D(a,\gamma)e^{-2|\gamma|^2/(s'-s)}
\end{aligned}
\tag{208}
$$

$$
\begin{aligned}
F_{\rho^2}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi}\, D(\nu,\mu/2)\Phi_\rho^{(-s)*}(\nu-\mu/2)\Phi_\rho^{(s)}(\nu+\mu/2) \\
&= \pi\int d^2\nu\, D(\nu,\mu)W_\rho^{(-s)}(\nu-\mu/4)W_\rho^{(s)}(\nu+\mu/4)
\end{aligned}
\tag{209}
$$

$$
\begin{aligned}
\Phi_{|\psi\rangle\langle\psi|}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi}\, D(\nu,\mu/2)\Phi_{|\psi\rangle\langle\psi|}^{(-s)*}(\nu-\mu/2)\Phi_{|\psi\rangle\langle\psi|}^{(s)}(\nu+\mu/2) \\
&= \pi\int d^2\nu\, D(\nu,\mu)W_{|\psi\rangle\langle\psi|}^{(-s)}(\nu-\mu/4)W_{|\psi\rangle\langle\psi|}^{(s)}(\nu+\mu/4)
\end{aligned}
\tag{210}
$$

$$
s=+1\text{:}\ \begin{cases}
\dfrac{1}{\pi}\tilde F_\rho^{(+1)}(\beta) = W_\rho^{(+1)}(\beta) \equiv P(\beta) = \begin{pmatrix}\text{Glauber} \\ P\text{ function}\end{pmatrix} \\
\rho = \displaystyle\int d^2\beta\, \tilde D^{(-1)}(a,\beta)P(\beta) = \int d^2\beta\, P(\beta)|\beta\rangle\langle\beta| \\
P(\beta) = \displaystyle\int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(+1)}(\alpha)D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi^2}\,\mathrm{tr}\big(\rho e^{\alpha a^\dagger}e^{-\alpha^* a}\big)D(\alpha,\beta)
\end{cases}
\tag{211}
$$

$$
s=0\text{:}\ \begin{cases}
\dfrac{1}{\pi}\tilde F_\rho^{(0)}(\beta) = W_\rho^{(0)}(\beta) \equiv W(\beta) = \begin{pmatrix}\text{Wigner} \\ \text{function}\end{pmatrix} \\
\rho = \displaystyle\int d^2\beta\, \tilde D^{(0)}(a,\beta)W(\beta) = 2P\int d^2\beta\, D(a,-2\beta)W(\beta) \\
W(\beta) = \displaystyle\int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(0)}(\alpha)D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi^2}\,\mathrm{tr}\big(\rho D(a,\alpha)\big)D(\alpha,\beta) \\
W(\beta) = \dfrac{1}{\pi}\mathrm{tr}\big(\rho\tilde D^{(0)}(a,\beta)\big) = \underbrace{\dfrac{2}{\pi}\mathrm{tr}\big(\rho D(a,\beta)PD^\dagger(a,\beta)\big)}_{\le 2/\pi} = \dfrac{2}{\pi}\mathrm{tr}\big(\rho PD(a,-2\beta)\big) \\
\qquad\quad = \dfrac{2}{\pi^2}\displaystyle\int d^2\alpha\,\langle\beta-\alpha|\rho|\beta+\alpha\rangle D(\alpha,\beta) \\
\mathrm{tr}(\rho_1\rho_2) = \pi\displaystyle\int d^2\beta\, W_{\rho_1}(\beta)W_{\rho_2}(\beta) \qquad \mathrm{tr}(\rho^2) = \pi\int d^2\beta\, W^2(\beta)
\end{cases}
\tag{212}
$$

If position β_1 and momentum β_2 are used as the variables in the Wigner function, it is conventional to use a rescaled Wigner function defined by

$$
W'(\beta_1,\beta_2) = \frac{1}{2}W(\beta) = \int \frac{d\alpha_1\, d\alpha_2}{(2\pi)^2}\,\mathrm{tr}\big(\rho e^{i(\alpha_2 x-\alpha_1 p)}\big)e^{i(\beta_2\alpha_1-\beta_1\alpha_2)}\ .
\tag{213}
$$

$$
\begin{aligned}
\langle x|\rho|x'\rangle &= \int \frac{d\beta_2}{2}\, W\bigg(\frac{1}{\sqrt2}\Big((x+x')/2+i\beta_2\Big)\bigg)\, e^{i\beta_2(x-x')} \\
&= \int d\beta_2\, W'\big((x+x')/2,\beta_2\big)e^{i\beta_2(x-x')} \\
W'(\beta_1,\beta_2) &= \frac{1}{2}W(\beta) = \int \frac{d\xi}{2\pi}\,\langle\beta_1+\xi/2|\rho|\beta_1-\xi/2\rangle e^{-i\beta_2\xi}
\end{aligned}
\tag{214}
$$

$$\langle p|\rho|p'\rangle = \int \frac{d\beta_1}{2}\, W\biggl(\frac{1}{\sqrt{2}}\Bigl(\beta_1 + i(p+p')/2\Bigr)\biggr)\, e^{-i\beta_1(p-p')}$$

$$= \int d\beta_1\, W'\bigl(\beta_1, (p+p')/2\bigr)e^{-i\beta_1(p-p')} \tag{215}$$

$$W'(\beta_1,\beta_2) = \frac{1}{2}W(\beta) = \int \frac{d\eta}{2\pi}\, \langle\beta_2+\eta/2|\rho|\beta_2-\eta/2\rangle e^{i\beta_1\eta}$$

$$1 = \mathrm{tr}(\rho) = \int d^2\beta\, W(\beta) = \int d\beta_1\, d\beta_2\, W'(\beta_1,\beta_2) \tag{216}$$

$$\mathrm{tr}(\rho_1\rho_2) = 2\pi \int d\beta_1\, d\beta_2\, W'_{\rho_1}(\beta_1,\beta_2)W'_{\rho_2}(\beta_1,\beta_2) \qquad \mathrm{tr}(\rho^2) = 2\pi \int d\beta_1\, d\beta_2\, W'^2(\beta_1,\beta_2) \tag{217}$$

$$s=-1\colon \begin{cases} \dfrac{1}{\pi}\tilde{F}^{(-1)}_\rho(\beta) = W^{(-1)}_\rho(\beta) \equiv Q(\beta) = \begin{pmatrix}\text{Husimi}\\ Q\text{ function}\end{pmatrix} \\ \rho = \displaystyle\int d^2\beta\, \tilde{D}^{(+1)}(a,\beta)Q(\beta) \\ Q(\beta) = \dfrac{1}{\pi}\mathrm{tr}\bigl(\rho\tilde{D}^{(-1)}(a,\beta)\bigr) = \dfrac{1}{\pi}\langle\beta|\rho|\beta\rangle \le \dfrac{1}{\pi} \end{cases} \tag{218}$$

$$\Phi^{(s)}_{|\gamma\rangle\langle\gamma|}(\alpha) = \langle\gamma|D^{(s)}(a,\alpha)|\gamma\rangle = e^{(s-1)|\alpha|^2/2}D(\gamma,\alpha) \tag{219}$$

$$W^{(s)}_{|\gamma\rangle\langle\gamma|}(\beta) = \int \frac{d^2\alpha}{\pi^2}\, \Phi^{(s)}_{|\gamma\rangle\langle\gamma|}(\alpha)D(\alpha,\beta)$$

$$= \int \frac{d^2\alpha}{\pi^2}\, e^{(s-1)|\alpha|^2/2}D(\alpha,\beta-\gamma) \tag{220}$$

$$= \frac{2}{\pi(1-s)}e^{-2|\beta-\gamma|^2/(1-s)}$$

$$\rho = |\gamma\rangle\langle\gamma|\colon \begin{cases} s=+1\colon \quad P(\beta) = \delta^2(\beta-\gamma) \\ s=0\colon \quad W(\beta) = \dfrac{2}{\pi}\, e^{-2|\beta-\gamma|^2}\ , \quad W'(\beta_1,\beta_2) = \dfrac{1}{\pi}\, e^{-(\beta_1-\gamma_1)^2-(\beta_2-\gamma_2)^2} \\ s=-1\colon \quad Q(\beta) = \dfrac{1}{\pi}\, e^{-|\beta-\gamma|^2} = \dfrac{1}{\pi}\, |\langle\beta|\gamma\rangle|^2 \end{cases} \tag{221}$$

$$\rho = |\gamma\rangle\langle\gamma|: \quad W(\beta) = \frac{2}{\pi}\langle\gamma|D^\dagger(a,-\beta)PD(a,-\beta)|\gamma\rangle$$

$$= \frac{2}{\pi}\langle\gamma-\beta|P|\gamma-\beta\rangle$$

$$= \frac{2}{\pi}\langle\gamma-\beta|\beta-\gamma\rangle \tag{222}$$

$$= \frac{2}{\pi}\, e^{-2|\beta-\gamma|^2}$$

The remainder of this section deals with number-state projectors, i.e., $\rho = |n\rangle\langle n|$.

$$\Phi^{(s)}(\alpha) = e^{s|\alpha|^2/2}\langle n|D(a,\alpha)|n\rangle = e^{(s-1)|\alpha|^2/2}L_n\bigl(|\alpha|^2\bigr) \tag{223}$$

$$\begin{aligned}
W^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi^2}\, e^{(s-1)|\alpha|^2/2} L_n\big(|\alpha|^2\big) D(\alpha,\beta) \\
(\beta=\beta^*) \quad &= \frac{1}{\pi}\int_0^\infty dx\, e^{-(1-s)x/2} L_n(x) \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-2i\beta\sqrt{x}\sin\phi} \\
&= \frac{1}{\pi}\int_0^\infty dx\, e^{-(1-s)x/2} L_n(x) \frac{1}{\pi}\underbrace{\int_0^{\pi} d\phi\, \cos(2\beta\sqrt{x}\sin\phi)}_{=\pi J_0(2\beta\sqrt{x})} \\
(\beta \text{ general}) \quad &= \frac{1}{\pi}\int_0^\infty dx\, e^{-(1-s)x/2} L_n(x) J_0\big(2|\beta|\sqrt{x}\big)
\end{aligned} \tag{224}$$

$$\begin{aligned}
W^{(s)}(\beta) &= \frac{1}{\pi}\int_0^\infty dx\, e^{-(1-s)x/2} L_n(x) J_0\big(2|\beta|\sqrt{x}\big) \\
&= \frac{1}{\pi}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{(-1)^k}{k!}\int_0^\infty dx\, e^{-(1-s)x/2} x^k J_0\big(2|\beta|\sqrt{x}\big) \\
&= \frac{2}{\pi(1-s)}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{k!}\left(\frac{2}{s-1}\right)^k \underbrace{\int_0^\infty dt\, e^{-t} t^k J_0\left(2|\beta|\sqrt{\frac{2}{1-s}t}\right)}_{=k!\,e^{-2|\beta|^2/(1-s)} L_k\left(\frac{2}{1-s}|\beta|^2\right)} \\
&= \frac{2}{\pi(1-s)} e^{-2|\beta|^2/(1-s)} \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{2}{s-1}\right)^k L_k\left(\frac{2}{1-s}|\beta|^2\right) \\
&= \frac{2}{\pi(1-s)}\left(\frac{s+1}{s-1}\right)^n e^{-2|\beta|^2/(1-s)} L_n\left(\frac{4}{1-s^2}|\beta|^2\right)
\end{aligned} \tag{225}$$

Equation (225) uses

$$k!\, e^{-x} L_k(x) = \int_0^\infty dt\, e^{-t} t^k J_0\big(2\sqrt{tx}\big) \qquad \text{(A\&S 22.10.14)}$$

and

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mu^k L_k(x) = (1+\mu)^n L_n\left(\frac{\mu}{1+\mu}x\right) \qquad \text{(A\&S 22.12.7)}$$

$$\begin{aligned}
s=0:\quad W(\beta) &= \frac{2}{\pi}\langle n|PD(a,-2\beta)|n\rangle = \frac{2}{\pi}(-1)^n e^{-2|\beta|^2} L_n\big(4|\beta|^2\big) \\
W'(\beta_1,\beta_2) &= \frac{(-1)^n}{\pi}\, e^{-(\beta_1^2+\beta_2^2)} L_n\big(2(\beta_1^2+\beta_2^2)\big) \\
(-1)^n e^{-|\alpha|^2} L_n\big(2|\alpha|^2\big) &= \int \frac{d^2\beta}{\pi}\, e^{-|\beta|^2} L_n\big(2|\beta|^2\big) D(\beta,\alpha)
\end{aligned} \tag{226}$$

$$\begin{aligned}
s=-1:\quad Q(\beta) &= \frac{1}{\pi}\,|\langle\beta|n\rangle|^2 = \frac{1}{\pi}\, e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!} \\
e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!} &= \int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} L_n\big(|\alpha|^2\big) D(\alpha,\beta) \\
e^{-|\alpha|^2} L_n\big(|\alpha|^2\big) &= \int \frac{d^2\beta}{\pi}\, e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!} D(\beta,\alpha)
\end{aligned} \tag{227}$$

$$\langle n|[(a^\dagger)^k a^k]_{(s)}|n\rangle = \left.\frac{\partial^{2k}\Phi^{(s)}(\alpha)}{\partial\alpha^k\partial(-\alpha^*)^k}\right|_{\alpha=0} = (-1)^k k!\,\left.\frac{d^k}{dx^k}e^{(s-1)x/2}L_n(x)\right|_{x=0}$$
$$= \int d^2\beta\,|\beta|^{2k}W^{(s)}(\beta) = \pi\int_0^\infty dx\,x^k W^{(s)}(x) \tag{228}$$
$$W^{(s)}(\beta) \to W^{(s)}(|\beta|^2) = W^{(s)}(x)$$

$$(-1)^k k! L_n^{(k)}(0) = \langle n|[(a^\dagger)^k a^k]_{(+1)}|n\rangle = \langle n|(a^\dagger)^k a^k|n\rangle = (-1)^k(-n)_k = \begin{cases} n!/(n-k)!\,, & k\le n \\ 0\,, & k>n \end{cases}$$
$$L_n(x) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{(-x)^k}{k!} = {}_1F_1(-n;1;x) = M(-n,1,x) \tag{229}$$

$$\langle n|[(a^\dagger)^k a^k]_{(-1)}|n\rangle = \langle n|a^k(a^\dagger)^k|n\rangle$$
$$= (-1)^k k!\,\left.\frac{d^k}{dx^k}e^{-x}L_n(x)\right|_{x=0}$$
$$= \frac{1}{n!}\int_0^\infty dx\,x^{n+k}e^{-x} \tag{230}$$
$$= \frac{(n+k)!}{n!} = (n+1)_k$$
$$L_n(x) = e^x\sum_{k=0}^{\infty}\frac{(n+k)!}{n!\,k!}\frac{(-x)^k}{k!}$$

$$\langle n|[(a^\dagger)^k a^k]_{(s)}|n\rangle = (-1)^k k!\,\left.\frac{d^k}{dx^k}e^{(s-1)x/2}L_n(x)\right|_{x=0}$$
$$= \left(\frac{1-s}{2}\right)^k k!\underbrace{\sum_{m=0}^{\min(n,k)}\frac{k!}{m!(k-m)!}\frac{n!}{m!(n-m)!}\left(\frac{2}{1-s}\right)^m}_{= {}_2F_1\left(-n,-k;1;\frac{2}{1-s}\right)} \tag{231}$$
$$= \left(\frac{1-s}{2}\right)^k k!\,{}_2F_1\left(-n,-k;1;\frac{2}{1-s}\right)$$
$$= \left(\frac{1-s}{2}\right)^k\left(\frac{s+1}{s-1}\right)^n k!\,{}_2F_1\left(-n,k+1;1;\frac{2}{1+s}\right) \qquad \text{(A\&S 15.3.4)}$$

$$\langle n|[(a^\dagger)^k a^k]_{(0)}|n\rangle = 2(-1)^n\int_0^\infty dx\,x^k e^{-2x}L_n(4x)$$
$$= \frac{(-1)^n}{2^k}\int_0^\infty du\,u^k e^{-u}L_n(2u)$$
$$= \frac{(-1)^n}{2^k}\underbrace{\sum_{m=0}^{n}\frac{(m+k)!}{m!}\frac{n!}{m!(n-m)!}(-2)^m}_{= k!\,{}_2F_1(-n,k+1;1;2)} \tag{232}$$
$$= \frac{k!}{2^k}(-1)^n{}_2F_1(-n,k+1;1;2)$$

The sums in Eqs. (231) and (232) are related by

$$
\begin{aligned}
\sum_{m=0}^{n}\frac{(m+k)!}{m!}\frac{n!}{m!(n-m)!}x^m &= \frac{d^k}{dx^k}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}x^{m+k}\\
&= \frac{d^k}{dx^k}x^k(1+x)^n\\
&= \sum_{m=0}^{\min(n,k)}\frac{k!}{m!(k-m)!}\frac{d^{k-m}x^k}{dx^{k-m}}\frac{d^m(1+x)^n}{dx^m}\\
&= k!\sum_{m=0}^{\min(n,k)}\frac{k!}{m!(k-m)!}\frac{n!}{m!(n-m)!}x^m(1+x)^{n-m}
\end{aligned}
\tag{233}
$$

$$
\begin{aligned}
\frac{1}{k!}\int_0^\infty du\,u^k e^{-u}L_n(cu) &= \sum_{m=0}^{n}\frac{(k+m)!}{k!\,m!}\frac{n!}{m!(n-m)!}(-c)^m = {}_2F_1(-n,k+1;1;c)\\
&= (1-c)^n\sum_{m=0}^{\min(n,k)}\frac{k!}{m!(k-m)!}\frac{n!}{m!(n-m)!}\left(\frac{c}{c-1}\right)^m
\end{aligned}
\tag{234}
$$

$$
\begin{aligned}
[(a^\dagger)^k a^k]_{(s)} &= \left(\frac{1-s}{2}\right)^k\left(\frac{s+1}{s-1}\right)^{a^\dagger a}k!\,{}_2F_1\!\left(-a^\dagger a,k+1;1;\frac{2}{1+s}\right)\\
&= \sum_{m=0}^{k}\frac{1}{(k-m)!}\left(\frac{1-s}{2}\right)^{k-m}\left(\frac{k!}{m!}\right)^2(a^\dagger a)^{(m)}\\
&= \sum_{m=0}^{k}\frac{1}{m!}\left(\frac{1-s}{2}\right)^{m}\left(\frac{k!}{(k-m)!}\right)^2(a^\dagger a)^{k-m}
\end{aligned}
\tag{235}
$$

$$
\int\frac{d^2\alpha}{\pi}\,e^{-(1+s)|\alpha|^2/2}L_n\big(|\alpha|^2\big)D(\alpha,\beta) = \frac{2}{1+s}\left(\frac{1-s}{1+s}\right)^n e^{-2|\beta|^2/(1+s)}(-1)^nL_n\!\left(\frac{4}{1-s^2}|\beta|^2\right)
\tag{236}
$$

$$
\begin{aligned}
\int\frac{d^2\alpha}{\pi}(-\alpha)^k(\alpha^*)^l e^{-(1+s)|\alpha|^2/2}L_n\big(|\alpha|^2\big) &= \delta_{kl}(-1)^k\underbrace{\int\frac{d^2\alpha}{\pi}|\alpha|^{2k}e^{-(1+s)|\alpha|^2/2}L_n\big(|\alpha|^2\big)}_{=\int_0^\infty dx\,x^k e^{-(1+s)x/2}L_n(x)}\\
&= \delta_{kl}(-1)^k\left(\frac{2}{1+s}\right)^{k+1}\underbrace{\int_0^\infty du\,u^k e^{-u}L_n\!\left(\frac{2}{1+s}u\right)}_{=k!\,{}_2F_1(-n,k+1;1;2/(1+s))}\\
&= \delta_{kl}(-1)^k k!\left(\frac{2}{1+s}\right)^{k+1}{}_2F_1\big(-n,k+1;1;2/(1+s)\big)
\end{aligned}
\tag{237}
$$

$$
\begin{aligned}
&\int \frac{d^2\alpha}{\pi}\, e^{-(1+s)|\alpha|^2/2} L_n\big(|\alpha|^2\big) D(\alpha,\beta) \\
&\qquad = \sum_{k,l} \frac{(\beta^*)^k \beta^l}{k!\,l!} \int \frac{d^2\alpha}{\pi} (-\alpha)^k (\alpha^*)^l e^{-(1+s)|\alpha|^2/2} L_n\big(|\alpha|^2\big) \\
&\qquad = \sum_{k=0}^{\infty} \frac{(-|\beta|^2)^k}{k!} \left(\frac{2}{1+s}\right)^{k+1} \underbrace{{}_2F_1\big(-n, k+1; 1; 2/(1+s)\big)}_{} \\
&\qquad\qquad\qquad\qquad = \left(\frac{s-1}{s+1}\right)^n \left(\frac{2}{1-s}\right)^k (-1)^k \left.\frac{d^k}{dx^k} e^{-(1-s)x/2} L_n(x)\right|_{x=0} \\
&\qquad = (-1)^n \frac{2}{1+s} \left(\frac{1-s}{1+s}\right)^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{4}{1-s^2}|\beta|^2\right)^k \left.\frac{d^k}{dx^k} e^{-(1-s)x/2} L_n(x)\right|_{x=0} \\
&\qquad = \frac{2}{1+s} \left(\frac{1-s}{1+s}\right)^n e^{-2|\beta|^2/(1+s)} (-1)^n L_n\left(\frac{4}{1-s^2}|\beta|^2\right)
\end{aligned}
\tag{238}
$$

$$
\begin{aligned}
|n\rangle\langle n| &= \int \frac{d^2\alpha}{\pi}\, D^{(s)\dagger}(a,\alpha) \langle n| D^{(-s)}(a,\alpha) |n\rangle \\
&= \sum_{k,l} \frac{[(a^\dagger)^k a^l]_{(s)}}{k!\,l!} \int \frac{d^2\alpha}{\pi} (-\alpha)^k (\alpha^*)^l e^{-(1+s)|\alpha|^2/2} L_n\big(|\alpha|^2\big) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{2}{1+s}\right)^{k+1} {}_2F_1\left(-n, k+1; 1; \frac{2}{1+s}\right) [(a^\dagger)^k a^k]_{(s)}
\end{aligned}
\tag{239}
$$

15. Thermal states

$$\rho = \frac{1}{Z} e^{-\mu a^\dagger a} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\mu n} |n\rangle\langle n| \,, \qquad \mu = \beta\hbar\omega = \hbar\omega/kT \tag{240}$$

$$Z = \operatorname{tr}\big(e^{-\mu a^\dagger a}\big) = \sum_{n=0}^{\infty} e^{-\mu n} = \frac{1}{1-e^{-\mu}} \tag{241}$$

$$\bar{n} = \langle a^\dagger a\rangle = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-\mu n} = -\frac{1}{Z}\frac{\partial Z}{\partial \mu} = \frac{1}{e^{\mu}-1} \tag{242}$$

$$\frac{d\bar{n}}{d(-\mu)} = \bar{n}(\bar{n}+1) \tag{243}$$

$$\begin{aligned} e^{\mu} &= \frac{1+\bar{n}}{\bar{n}} = \frac{Z}{Z-1} \\ Z &= 1+\bar{n} = \frac{1}{1-e^{-\mu}} \\ \rho &= \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}}\right)^{a^\dagger a} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^{n} |n\rangle\langle n| \end{aligned} \tag{244}$$

$$\begin{aligned} \overline{n^2} &= \langle (a^\dagger a)^2\rangle = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 e^{-\mu n} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \mu^2} = \frac{1}{e^{\mu}-1} + \frac{2}{(e^{\mu}-1)^2} = \bar{n} + 2\bar{n}^2 \\ (\Delta n)^2 &= \overline{n^2} - \bar{n}^2 = \bar{n}(\bar{n}+1) \end{aligned} \tag{245}$$

$$\overline{n^k} = \langle (a^\dagger a)^k\rangle = \frac{1}{Z}\sum_{n=0}^{\infty} n^k e^{-\mu n} = \frac{1}{Z}\frac{\partial^k Z}{\partial (-\mu)^k} = \frac{1}{\bar{n}+1}\left(\bar{n}(\bar{n}+1)\frac{\partial}{\partial \bar{n}}\right)^{k} (\bar{n}+1) \tag{246}$$

$$\overline{n^k} = \frac{1}{Z}\frac{\partial}{\partial(-\mu)} Z\overline{n^{k-1}} = \bar{n}\frac{\partial}{\partial \bar{n}}(\bar{n}+1)\overline{n^{k-1}} \tag{247}$$

$$S = -\operatorname{tr}(\rho\ln\rho) = \ln Z + \mu\bar{n} = (1+\bar{n})\ln(1+\bar{n}) - \bar{n}\ln\bar{n} \tag{248}$$

$$\begin{aligned} \Big\langle n\Big|\Big(\int \frac{d^2\beta}{\pi\bar{n}}\, e^{-|\beta|^2/\bar{n}} |\beta\rangle\langle\beta|\Big)\Big|m\Big\rangle &= \int \frac{d^2\beta}{\pi\bar{n}}\, e^{-|\beta|^2/\bar{n}} \langle n|\beta\rangle\langle\beta|m\rangle \\ &= \frac{1}{\sqrt{n!\,m!}} \int \frac{d^2\beta}{\pi\bar{n}}\, e^{-|\beta|^2(1+\bar{n})/\bar{n}} \beta^n (\beta^*)^m \\ &= \frac{1}{2\sqrt{n!\,m!}} \int \frac{d|\beta|^2\, d\phi}{\pi\bar{n}}\, e^{-|\beta|^2(1+\bar{n})/\bar{n}} |\beta|^{n+m} e^{i(n-m)\phi} \\ &= \frac{1}{n!}\delta_{nm} \int_0^\infty \frac{d|\beta|^2}{\bar{n}}\, e^{-|\beta|^2(1+\bar{n})/\bar{n}} |\beta|^{2n} \\ &= \frac{1}{n!}\delta_{nm} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \int_0^\infty du\, e^{-u} u^n \\ &= \delta_{nm}\frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \end{aligned} \tag{249}$$

$$\rho = \int \frac{d^2\beta}{\pi\bar{n}}\, e^{-|\beta|^2/\bar{n}} |\beta\rangle\langle\beta| \quad\Longrightarrow\quad P(\beta) = W_\rho^{(+1)}(\beta) = \frac{1}{\pi}\tilde{F}_\rho^{(+1)}(\beta) = \frac{1}{\pi\bar{n}} e^{-|\beta|^2/\bar{n}} \tag{250}$$

$$\Phi_\rho^{(+1)}(\alpha) = F_\rho^{(+1)}(\alpha) = \int d^2\beta\, P(\beta) D(\beta,\alpha) = \int \frac{d^2\beta}{\pi\bar{n}}\, e^{-|\beta|^2/\bar{n}} D(\beta,\alpha) = e^{-\bar{n}|\alpha|^2} \tag{251}$$

$$\Phi_\rho^{(s)}(\alpha) = e^{(s-1)|\alpha|^2/2}\Phi_\rho^{(+1)}(\alpha) = e^{-[\bar{n}+(1-s)/2]|\alpha|^2} \tag{252}$$

$$\left\langle [(a^\dagger)^k a^l]_{(s)} \right\rangle = \left.\frac{\partial^{k+l}\Phi_\rho^{(s)}(\alpha)}{\partial\alpha^k\,\partial(-\alpha^*)^l}\right|_{\alpha=0} = k!\left(\bar{n}+\frac{1-s}{2}\right)^k \delta_{kl} \tag{253}$$

$$\langle {:}(a^\dagger a)^k{:}\rangle = \langle (a^\dagger a)^{(k)}\rangle = \langle (a^\dagger)^k a^k\rangle = k!\,\bar{n}^k \tag{254}$$

$$\begin{aligned}\frac{1}{\pi}\tilde{F}_\rho^{(s)}(\beta) = W_\rho^{(s)}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(s)}(\alpha)D(\alpha,\beta)\\ &= \int \frac{d^2\alpha}{\pi^2}\,e^{-[\bar{n}+(1-s)/2]|\alpha|^2}D(\alpha,\beta)\\ &= \frac{1}{\pi[\bar{n}+(1-s)/2]}e^{-|\beta|^2/[\bar{n}+(1-s)/2]}\end{aligned} \tag{255}$$

$$\begin{aligned}e^{-\bar{n}|\alpha|^2} &= \Phi_\rho^{(+1)}(\alpha)\\ &= \mathrm{tr}\big(\rho D^{(+1)}(a,\alpha)\big)\\ &= \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \langle n|D^{(+1)}(a,\alpha)|n\rangle\\ &= \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n L_n\big(|\alpha|^2\big)\end{aligned} \tag{256}$$

$$e^{-\mu a^\dagger a} = {:}e^{-(1-e^{-\mu})a^\dagger a}{:} = {:}e^{-a^\dagger a/Z}{:} = {:}e^{-a^\dagger a/(1+\bar{n})}{:} \tag{257}$$

$$\langle e^{-\lambda a^\dagger a}\rangle = \langle {:}e^{(e^{-\lambda}-1)a^\dagger a}{:}\rangle = \sum_{k=0}^{\infty}\frac{1}{k!}(e^{-\lambda}-1)^k\langle (a^\dagger)^k a^k\rangle = \sum_{k=0}^{\infty}\big(\bar{n}(e^{-\lambda}-1)\big)^k = \frac{1}{1+\bar{n}(1-e^{-\lambda})} \tag{258}$$

$$\langle (1-\lambda)^{a^\dagger a}\rangle = \frac{1}{1+\bar{n}}\sum_{k=0}^{\infty}\left((1-\lambda)\frac{\bar{n}}{1+\bar{n}}\right)^k = \frac{1}{1+\lambda\bar{n}} \tag{259}$$

$$\overline{n^k} = \langle (a^\dagger a)^k\rangle = \left.\frac{\partial^k\langle e^{\lambda a^\dagger a}\rangle}{\partial\lambda^k}\right|_{\lambda=0} = \left.\frac{\partial^k}{\partial\lambda^k}\left(\frac{1}{1-\bar{n}(e^\lambda-1)}\right)\right|_{\lambda=0} \tag{260}$$

$$\begin{aligned}P(\beta) &= W_\rho^{(+1)}(\beta) = \frac{1}{\pi\bar{n}}e^{-|\beta|^2/\bar{n}}\\ W(\beta) &= W_\rho^{(0)}(\beta) = \frac{1}{\pi(\bar{n}+1/2)}e^{-|\beta|^2/(\bar{n}+1/2)}\\ Q(\beta) &= W_\rho^{(-1)}(\beta) = \frac{1}{\pi}\langle\beta|\rho|\beta\rangle = \frac{1}{\pi(\bar{n}+1)}\langle\beta|{:}e^{-a^\dagger a/(\bar{n}+1)}{:}|\beta\rangle = \frac{1}{\pi(\bar{n}+1)}e^{-|\beta|^2/(\bar{n}+1)}\end{aligned} \tag{261}$$

$$\mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = \int \frac{d^2\alpha}{\pi}\,e^{s|\alpha|^2}D^\dagger(a,\alpha)e^{-\mu a^\dagger a}D(a,\alpha) \tag{262}$$

$$
\begin{aligned}
\langle\beta|\mathcal{G}^{(s)}(e^{-\mu a^\dagger a})|\beta\rangle &= \int \frac{d^2\alpha}{\pi}\, e^{s|\alpha|^2} \langle\beta|D^\dagger(a,\alpha)e^{-\mu a^\dagger a}D(a,\alpha)|\beta\rangle \qquad e^{-\mu} = \frac{\bar{n}}{1+\bar{n}} \\
&= \int \frac{d^2\alpha}{\pi}\, e^{s|\alpha|^2} \underbrace{\langle\beta+\alpha|e^{-\mu a^\dagger a}|\beta+\alpha\rangle}_{= e^{-|\beta+\alpha|^2/(1+\bar{n})}} \\
&= e^{-|\beta|^2/(1+\bar{n})} \underbrace{\int \frac{d^2\alpha}{\pi} \exp\!\left(-\frac{|\alpha|^2}{1+\bar{n}}(1-s(1+\bar{n}))\right)\exp\!\left(-\frac{\alpha\beta^*+\alpha^*\beta}{1+\bar{n}}\right)}_{= \frac{1+\bar{n}}{1-s(1+\bar{n})}\exp\left(\frac{|\beta|^2}{1+\bar{n}}\frac{1}{1-s(1+\bar{n})}\right), \quad s(1+\bar{n}) \le 1} \\
&= \frac{1+\bar{n}}{1-s(1+\bar{n})} \underbrace{\exp\!\left[-|\beta|^2\left(-\frac{s}{1-s(1+\bar{n})}\right)\right]}_{= \langle\beta|e^{-\nu a^\dagger a}|\beta\rangle} \qquad e^{-\nu} = \frac{1-s\bar{n}}{1-s(1+\bar{n})} = \frac{\bar{n}-1/s}{\bar{n}+1-1/s} \\
&= \frac{1+\bar{n}}{1-s(1+\bar{n})}\langle\beta|e^{-\nu a^\dagger a}|\beta\rangle \\
&= -\frac{1}{s}\frac{\bar{n}+1}{\bar{n}+1-1/s}\langle\beta|e^{-\nu a^\dagger a}|\beta\rangle
\end{aligned}
\tag{263}
$$

$$
\mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = \frac{1+\bar{n}}{1-s(1+\bar{n})}e^{-\nu a^\dagger a} = -\frac{1}{s}\frac{\bar{n}+1}{\bar{n}+1-1/s}\left(\frac{\bar{n}-1/s}{\bar{n}+1-1/s}\right)^{a^\dagger a} \tag{264}
$$

$$
\mathcal{G}^{(s)}\!\left(\left(\frac{s'+1}{s'-1}\right)^{a^\dagger a}\right) = -\frac{1}{s}\frac{1-s'}{1-s'-2/s}\left(\frac{s'+2/s+1}{s'+2/s-1}\right)^{a^\dagger a} \tag{265}
$$

$$
\begin{aligned}
s \to 0^- \;\; (\nu = -s)\!: \quad &\mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = (1+\bar{n})e^{s(1+\bar{n})}e^{s a^\dagger a} \\
&\mathcal{G}^{(0)}(e^{-\mu a^\dagger a}) = (1+\bar{n})1 = Z1
\end{aligned}
\tag{266}
$$

$$
\mathcal{G}^{(-1)}(e^{-\mu a^\dagger a}) = \left(\frac{\bar{n}+1}{\bar{n}+2}\right)^{a^\dagger a+1} \tag{267}
$$

$$
s \to -\infty\!: \quad \mathcal{G}^{(s)}(e^{-\mu a^\dagger a}) = -\frac{1}{s}e^{-\mu a^\dagger a} \tag{268}
$$

16. Single-mode squeeze operator and single-mode squeezed states

For additional information, see C. M. Caves and B. L. Schumaker, *Physical Review A* **31**, 3068–3092 (1985) [CS]; B. L. Schumaker and C. M. Caves, *Physical Review A* **31**, 3093–3111 (1985) [SC]; and B. L. Schumaker, *Physics Reports* **135**(6), 317–408 (1986) [S].

$$S(r,\phi) \equiv \exp\Big(\tfrac{1}{2}r\big(a^2e^{-2i\phi} - (a^\dagger)^2 e^{2i\phi}\big)\Big) = \exp\Big(i\tfrac{1}{2}r\big[(xp+px)\cos 2\phi - (x^2-p^2)\sin 2\phi\big]\Big) \tag{269}$$

$$S(r,\phi+\pi) = S(r,\phi) \tag{270}$$

$$S^{-1}(r,\phi) = S^\dagger(r,\phi) = S(-r,\phi) = S(r,\phi+\pi/2) \tag{271}$$

$$e^{i\theta a^\dagger a} S(r,\phi) e^{-i\theta a^\dagger a} = S(r,\phi+\theta) \tag{272}$$

$$\begin{aligned} S(r,\phi) &= e^{-\Gamma A^\dagger} e^{-gB} e^{\Gamma^* A} = e^{-\Gamma A^\dagger} e^{\Gamma^* e^{2g} A} e^{-gB} = e^{-gB} e^{-\Gamma e^{2g} A^\dagger} e^{\Gamma^* A} \\ &= e^{\Gamma^* A} e^{-\Gamma e^{2g} A^\dagger} e^{gB} = e^{\Gamma^* A} e^{gB} e^{-\Gamma A^\dagger} = e^{gB} e^{\Gamma^* e^{2g} A} e^{-\Gamma A^\dagger} \\ A &\equiv \frac{1}{2}a^2 \qquad B \equiv a^\dagger a + \frac{1}{2} \qquad \Gamma \equiv e^{2i\phi}\tanh r \qquad g \equiv \ln(\cosh r) \end{aligned} \tag{273}$$

(See Appendix B of [SC].)

$$\begin{aligned} S^\dagger(r,\phi) a S(r,\phi) &= a\cosh r - a^\dagger e^{2i\phi}\sinh r \\ S^\dagger(r,\phi) a^\dagger S(r,\phi) &= a^\dagger \cosh r - a e^{-2i\phi}\sinh r \end{aligned} \tag{274}$$

$$S^\dagger(r,\phi) D(a,\alpha) S(r,\phi) = D(a\cosh r - a^\dagger e^{2i\phi}\sinh r, \alpha) = D(a, \alpha\cosh r + \alpha^* e^{2i\phi}\sinh r) \tag{275}$$

$$\begin{aligned} S^\dagger(r,\phi)(x\cos\phi + p\sin\phi)S(r,\phi) &= (x\cos\phi + p\sin\phi)e^{-r} \\ S^\dagger(r,\phi)(-x\sin\phi + p\cos\phi)S(r,\phi) &= (-x\sin\phi + p\cos\phi)e^{r} \end{aligned} \tag{276}$$

$$S^\dagger(r,0)xS(r,0) = xe^{-r} \qquad S^\dagger(r,0)pS(r,0) = p\,e^{r} \tag{277}$$

$$\begin{aligned} &S^\dagger(r',\phi')S(r,\phi) = e^{-i\Theta B} S(R,\Phi) = S(R,\Phi-\Theta)e^{-i\Theta B} \\ &B \equiv a^\dagger a + \frac{1}{2} \\ &e^{i\Theta}\cosh R \equiv \cosh r \cosh r' - e^{2i(\phi-\phi')}\sinh r \sinh r' \\ &e^{i(2\Phi-\Theta)}\sinh R \equiv e^{2i\phi}\sinh r\cosh r' - e^{2i\phi'}\cosh r \sinh r' \end{aligned} \tag{278}$$

(See Appendix B of [SC].)

$$S(r',\phi)S(r,\phi) = S(r+r',\phi) \tag{279}$$

$$\begin{aligned} |\alpha\rangle_{(r,\phi)} &\equiv D(a,\alpha)S(r,\phi)|0\rangle \\ &= D(a,\alpha)|0\rangle_{(r,\phi)} \\ &= S(r,\phi)D(a,\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)|0\rangle \\ &= S(r,\phi)|\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r\rangle \end{aligned} \tag{280}$$

$$e^{-i\theta a^\dagger a}|\alpha\rangle_{(r,\phi)} = |\alpha e^{-i\theta}\rangle_{(r,\phi-\theta)} \tag{281}$$

$$\begin{aligned} (a\cosh r + a^\dagger e^{2i\phi}\sinh r)|\alpha\rangle_{(r,\phi)} &= S(r,\phi)a|\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r\rangle \\ &= (\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)|\alpha\rangle_{(r,\phi)} \end{aligned} \tag{282}$$

$$S^\dagger(r,\phi)D^\dagger(a,\alpha)aD(a,\alpha)S(r,\phi) = a\cosh r - a^\dagger e^{2i\phi}\sinh r + \alpha \tag{283}$$

$$S^\dagger(r,\phi)D^\dagger(a,\gamma)D(a,\alpha)D(a,\gamma)S(r,\phi) = D(a\cosh r - a^\dagger e^{2i\phi}\sinh r + \gamma, \alpha) \\ = D(\gamma,\alpha)D(a, \alpha\cosh r + \alpha^* e^{2i\phi}\sinh r) \tag{284}$$

$$\begin{aligned} {}_{(r,\phi)}\langle\alpha|a|\alpha\rangle_{(r,\phi)} &= \alpha \\ {}_{(r,\phi)}\langle\alpha|a^2|\alpha\rangle_{(r,\phi)} &= \alpha^2 - e^{2i\phi}\cosh r\sinh r = \alpha^2 - \frac{1}{2}e^{2i\phi}\sinh 2r \\ {}_{(r,\phi)}\langle\alpha|a^\dagger a|\alpha\rangle_{(r,\phi)} &= |\alpha|^2 + \sinh^2 r = |\alpha|^2 + \frac{1}{2}(\cosh 2r - 1) \end{aligned} \tag{285}$$

$$\begin{aligned} {}_{(r,\phi)}\langle\alpha|x|\alpha\rangle_{(r,\phi)} &= \alpha_1 \\ {}_{(r,\phi)}\langle\alpha|p|\alpha\rangle_{(r,\phi)} &= \alpha_2 \\ {}_{(r,\phi)}\langle\alpha|(\Delta x)^2|\alpha\rangle_{(r,\phi)} &= \frac{1}{2}(\cosh 2r - \sinh 2r\cos 2\phi) = \frac{1}{2}(e^{-2r}\cos^2\phi + e^{2r}\sin^2\phi) \\ {}_{(r,\phi)}\langle\alpha|(\Delta p)^2|\alpha\rangle_{(r,\phi)} &= \frac{1}{2}(\cosh 2r + \sinh 2r\cos 2\phi) = \frac{1}{2}(e^{-2r}\sin^2\phi + e^{2r}\cos^2\phi) \\ {}_{(r,\phi)}\langle\alpha|\tfrac{1}{2}(\Delta x\Delta p + \Delta p\Delta x)|\alpha\rangle_{(r,\phi)} &= -\frac{1}{2}\sinh 2r\sin 2\phi \end{aligned} \tag{286}$$

$$\begin{aligned} |0\rangle_{(r,\phi)} = S(r,\phi)|0\rangle &= \frac{1}{\sqrt{\cosh r}}\exp\left(-\frac{1}{2}(a^\dagger)^2 e^{2i\phi}\tanh r\right)|0\rangle \\ &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!}\underbrace{(a^\dagger)^{2n}|0\rangle}_{=\sqrt{(2n)!}\,|2n\rangle} \\ &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{2}}e^{2i\phi}\tanh r\right)^n\sqrt{\frac{(2n-1)!!}{n!}}|2n\rangle \\ &\quad [(2n)! = 2^n n!\,(2n-1)!!] \end{aligned} \tag{287}$$

$${}_{(r,\phi)}\langle 0|0\rangle_{(r,\phi)} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n\frac{(2n-1)!!}{n!} = \frac{1}{\cosh r\sqrt{1-\tanh^2 r}} = 1 \tag{288}$$

$$\left(\text{Use}\quad (1+x)^{-1/2} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!}\left(\frac{-x}{2}\right)^n\quad .\right)$$

$$\begin{aligned} {}_{(r,\phi)}\langle 0|a^\dagger a|0\rangle_{(r,\phi)} &= \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n 2n\frac{(2n-1)!!}{n!} \\ &= \frac{\sinh^2 r}{\cosh^3 r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n\frac{(2n+1)!!}{n!} \\ &= \frac{\sinh^2 r}{\cosh^3 r}\frac{1}{(1-\tanh^2 r)^{3/2}} \\ &= \sinh^2 r \end{aligned} \tag{289}$$

$$\left(\text{Use}\quad (1+x)^{-3/2} = \sum_{n=0}^{\infty}\frac{(2n+1)!!}{n!}\left(\frac{-x}{2}\right)^n\quad .\right)$$

$$
\begin{aligned}
{}_{(r,\phi)}\langle 0|a^\dagger a(a^\dagger a-2)|0\rangle_{(r,\phi)} &= \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n 4n(n-1)\frac{(2n-1)!!}{n!} \\
&= \frac{\sinh^4 r}{\cosh^5 r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n \frac{(2n+3)!!}{n!} \\
&= \frac{\sinh^4 r}{\cosh^5 r}\frac{3}{(1-\tanh^2 r)^{5/2}} \\
&= 3\sinh^4 r
\end{aligned} \tag{290}
$$

$$
\left(\text{Use} \quad (1+x)^{-5/2} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(2n+3)!!}{n!}\left(\frac{-x}{2}\right)^n \quad .\right)
$$

$$
{}_{(r,\phi)}\langle 0|(a^\dagger a)^2|0\rangle_{(r,\phi)} = 3\sinh^4 r + 2\sinh^2 r \tag{291}
$$

$$
{}_{(r,\phi)}\langle 0|(\Delta a^\dagger a)^2|0\rangle_{(r,\phi)} = 2\sinh^4 r + 2\sinh^2 r = 2\sinh^2 r\cosh^2 r \tag{292}
$$

$$
\begin{aligned}
\langle x=0|0\rangle_{(r,\phi)} &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!}\underbrace{\sqrt{(2n)!}\langle x=0|2n\rangle}_{=\frac{1}{\pi^{1/4}2^n}H_{2n}(0)} \\
&= \frac{1}{\pi^{1/4}\sqrt{\cosh r}}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{2i\phi}\tanh r\right)^n\frac{(2n-1)!!}{n!} \\
&= \frac{1}{\pi^{1/4}}\frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}}
\end{aligned} \tag{293}
$$

[Use $H_{2n}(0) = (-1)^n(2n)!/n! = (-2)^n(2n-1)!!$: A&S 22.3.10.]

$$
\begin{aligned}
0 &= \langle x|(a\cosh r + a^\dagger e^{2i\phi}\sinh r)|0\rangle_{(r,\phi)} \\
&= \frac{1}{\sqrt{2}}\Big((\cosh r + e^{2i\phi}\sinh r)x + (\cosh r - e^{2i\phi}\sinh r)\frac{d}{dx}\Big)\langle x|0\rangle_{(r,\phi)} \\
\Longrightarrow \quad \langle x|0\rangle_{(r,\phi)} &= \frac{1}{\pi^{1/4}}\frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}}\exp\left(-\frac{1}{2}\frac{\cosh r + e^{2i\phi}\sinh r}{\cosh r - e^{2i\phi}\sinh r}x^2\right) \\
&= \frac{1}{\pi^{1/4}}\frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}}\exp\left(-\frac{1}{2}\frac{1+i\sinh 2r\sin 2\phi}{\cosh 2r - \sinh 2r\cos 2\phi}x^2\right)
\end{aligned} \tag{294}
$$

$$
\begin{aligned}
\langle x|\alpha\rangle_{(r,\phi)} &= \langle x|D(a,\alpha)|0\rangle_{(r,\phi)} \\
&= e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}\langle x-\alpha_1|0\rangle_{(r,\phi)} \\
&= \frac{1}{\pi^{1/4}}\frac{e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}}\exp\left(-\frac{1}{2}\frac{1+i\sinh 2r\sin 2\phi}{\cosh 2r - \sinh 2r\cos 2\phi}(x-\alpha_1)^2\right)
\end{aligned} \tag{295}
$$

$$
\begin{aligned}
\langle \beta|0\rangle_{(r,\phi)} &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!}\underbrace{\sqrt{(2n)!}\langle \beta|2n\rangle}_{=e^{-|\beta|^2/2}(\beta^*)^{2n}} \\
&= \frac{1}{\sqrt{\cosh r}}e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}(\beta^* e^{i\phi})^2\tanh r\right)^n}{n!} \\
&= \frac{1}{\sqrt{\cosh r}}e^{-|\beta|^2/2}\exp\left(-\frac{1}{2}(\beta^* e^{i\phi})^2\tanh r\right)
\end{aligned} \tag{296}
$$

$$\begin{aligned}
\langle\beta|\alpha\rangle_{(r,\phi)} &= \langle\beta|D(a,\alpha)|0\rangle_{(r,\phi)} \\
&= D(\beta,\alpha/2)\langle\beta-\alpha|0\rangle_{(r,\phi)} \\
&= \frac{1}{\sqrt{\cosh r}} D(\beta,\alpha/2)e^{-|\beta-\alpha|^2/2}\exp\biggl(-\frac{1}{2}[(\beta^*-\alpha^*)e^{i\phi}]^2\tanh r\biggr)
\end{aligned} \tag{297}$$

$$\begin{aligned}
|\langle\beta|\alpha\rangle_{(r,\phi)}|^2 &= |\langle\beta-\alpha|0\rangle_{(r,\phi)}|^2 \\
&= \frac{1}{\cosh r}\exp\biggl[-\biggl(|\beta-\alpha|^2+\frac{1}{2}\Bigl([(\beta-\alpha)e^{-i\phi}]^2+[(\beta^*-\alpha^*)e^{i\phi}]^2\Bigr)\tanh r\biggr)\biggr] \\
&= \frac{1}{\cosh r}\exp\biggl[-\frac{1}{2}\Bigl((\beta_1-\alpha_1)^2(1+\tanh r\cos 2\phi) \\
&\qquad + (\beta_2-\alpha_2)^2(1-\tanh r\cos 2\phi) \\
&\qquad + 2(\beta_1-\alpha_1)(\beta_2-\alpha_2)\tanh r\sin 2\phi\Bigr)\biggr]
\end{aligned} \tag{298}$$

$$\rho_{\alpha;r,\phi} \equiv |\alpha\rangle_{(r,\phi)\,(r,\phi)}\langle\alpha| \tag{299}$$

$$\begin{aligned}
\Phi^{(s)}_{\rho_{\gamma;r,\phi}}(\alpha) &= {}_{(r,\phi)}\langle\gamma|D^{(s)}(a,\alpha)|\gamma\rangle_{(r,\phi)} \\
&= e^{s|\alpha|^2/2}\langle 0|S^\dagger(r,\phi)D^\dagger(a,\gamma)D(a,\alpha)D(a,\gamma)S(r,\phi)|0\rangle \\
&= D(\gamma,\alpha)e^{s|\alpha|^2/2}\langle 0|D(a,\alpha\cosh r+\alpha^* e^{2i\phi}\sinh r)|0\rangle \\
&= D(\gamma,\alpha)e^{s|\alpha|^2/2}\exp\biggl(-\frac{1}{2}|\alpha\cosh r+\alpha^* e^{2i\phi}\sinh r|^2\biggr) \\
&= D(\gamma,\alpha)\exp\biggl[-\frac{1}{2}\biggl(|\alpha|^2(-s+\cosh 2r)+\frac{1}{2}[(\alpha e^{-i\phi})^2+(\alpha^* e^{i\phi})^2]\sinh 2r\biggr)\biggr] \\
&= D(\gamma,\alpha)\exp\biggl[-\frac{1}{4}\biggl(\alpha_1^2(-s+\cosh 2r+\sinh 2r\cos 2\phi) \\
&\qquad + \alpha_2^2(-s+\cosh 2r-\sinh 2r\cos 2\phi) \\
&\qquad + 2\alpha_1\alpha_2\sinh 2r\sin 2\phi\biggr)\biggr] \\
&= D(\gamma,\alpha)\exp\biggl[-\frac{1}{4}\biggl(\alpha_1^2(-s+e^{-2r}\sin^2\phi+e^{2r}\cos^2\phi) \\
&\qquad + \alpha_2^2(-s+e^{-2r}\cos^2\phi+e^{2r}\sin^2\phi) \\
&\qquad + 2\alpha_1\alpha_2\sinh 2r\sin 2\phi\biggr)\biggr]
\end{aligned} \tag{300}$$

$$\begin{aligned}
W^{(s)}_{\rho_{0;r,0}}(\beta) &= \int\frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{\rho_{0;r,0}}(\alpha)D(\alpha,\beta) \\
&= \frac{1}{\pi}\int\frac{d\alpha_1\,d\alpha_2}{2\pi}\,e^{i(\beta_2\alpha_1-\beta_1\alpha_2)}e^{-[\alpha_1^2(-s+e^{2r})+\alpha_2^2(-s+e^{-2r})]/4} \\
&= \frac{2}{\pi}\frac{1}{\sqrt{(-s+e^{-2r})(-s+e^{2r})}}\exp\biggl(-\frac{\beta_1^2}{-s+e^{-2r}}-\frac{\beta_2^2}{-s+e^{2r}}\biggr) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}}\exp\biggl(-\frac{2|\beta|^2(\cosh 2r-s)+[\beta^2+(\beta^*)^2]\sinh 2r}{1-2s\cosh 2r+s^2}\biggr)\;, \\
&\qquad s\le e^{-2|r|}
\end{aligned} \tag{301}$$

$$
\begin{aligned}
W^{(s)}_{\rho_{0;r,\phi}}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\, \Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi^2}\, \underbrace{\Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha e^{i\phi})}_{=\Phi^{(s)}_{\rho_{0;r,0}}(\alpha)} D(\alpha,\beta e^{-i\phi}) \\
&= W^{(s)}_{\rho_{0;r,0}}(\beta e^{-i\phi}) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(-\frac{2|\beta|^2(\cosh 2r - s) + [(\beta e^{-i\phi})^2 + (\beta^* e^{i\phi})^2]\sinh 2r}{1-2s\cosh 2r+s^2}\right) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(\frac{\begin{array}{c}-\beta_1^2(-s+\cosh 2r+\sinh 2r\cos 2\phi) \\ -\beta_2^2(-s+\cosh 2r-\sinh 2r\cos 2\phi) \\ -2\beta_1\beta_2\sinh 2r\sin 2\phi\end{array}}{1-2s\cosh 2r+s^2}\right), \quad s \le e^{-2|r|}
\end{aligned}
\tag{302}
$$

$$
\begin{aligned}
W^{(s)}_{\rho_{\gamma;r,\phi}}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\, \Phi^{(s)}_{\rho_{\gamma;r,\phi}}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi^2}\, \Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha) D(\alpha,\beta-\gamma) \\
&= W^{(s)}_{\rho_{0;r,\phi}}(\beta-\gamma) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(-\frac{2|\beta-\gamma|^2(\cosh 2r - s) + \left([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2\right)\sinh 2r}{1-2s\cosh 2r+s^2}\right) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(\frac{\begin{array}{c}-(\beta_1-\gamma_1)^2(-s+\cosh 2r+\sinh 2r\cos 2\phi) \\ -(\beta_2-\gamma_2)^2(-s+\cosh 2r-\sinh 2r\cos 2\phi) \\ -2(\beta_1-\gamma_1)(\beta_2-\gamma_2)\sinh 2r\sin 2\phi\end{array}}{1-2s\cosh 2r+s^2}\right), \quad s \le e^{-2|r|}
\end{aligned}
\tag{303}
$$

$$\rho = \rho_{\gamma;r,\phi}: \begin{cases} s=0: & W(\beta) = \frac{2}{\pi}\, e^{-2|\beta-\gamma|^2 \cosh 2r - ([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2)\sinh 2r} \\ & W'(\beta_1,\beta_2) = \frac{1}{\pi}\,\exp\begin{pmatrix} -(\beta_1-\gamma_1)^2(\cosh 2r + \sinh 2r\cos 2\phi) \\ -(\beta_2-\gamma_2)^2(\cosh 2r - \sinh 2r\cos 2\phi) \\ -2(\beta_1-\gamma_1)(\beta_2-\gamma_2)\sinh 2r\sin 2\phi \end{pmatrix} \\ & Q(\beta) = \frac{1}{\pi\cosh r} e^{-|\beta-\gamma|^2 - \frac{1}{2}([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2)\tanh r} \\ s=-1: & \quad = \frac{1}{\pi\cosh r}\exp\begin{pmatrix} -\frac{1}{2}(\beta_1-\gamma_1)^2(1+\tanh r\cos 2\phi) \\ -\frac{1}{2}(\beta_2-\gamma_2)^2(1-\tanh r\cos 2\phi) \\ -(\beta_1-\gamma_1)(\beta_2-\gamma_2)\tanh r\sin 2\phi \end{pmatrix} \\ & \quad = \frac{1}{\pi}\,|\langle\beta|\gamma\rangle_{(r,\phi)}|^2 \end{cases} \tag{304}$$

$$\begin{aligned} \rho = \rho_{0;r,0} \ : \quad W(\beta) &= \frac{2}{\pi}\ {}_{(r,0)}\langle 0|PD(a,-2\beta)|0\rangle_{(r,\phi)} \\ &= \frac{2}{\pi}\ {}_{(r,0)}\langle 0|D(a,-2\beta)|0\rangle_{(r,\phi)} \\ &= \frac{2}{\pi}\Phi^{(0)}_{\rho_{0;r,0}}(-2\beta) \\ &= \frac{2}{\pi} e^{-2|\beta|^2\cosh 2r - [\beta^2 + (\beta^*)^2]\sinh 2r} \end{aligned} \tag{305}$$

17. Auxiliary formulae

$$\frac{d^k f(x)g(x)}{dx^k} = \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \frac{d^m f(x)}{dx^m} \frac{d^{k-m} g(x)}{dx^{k-m}} \tag{306}$$

$$\frac{\partial^{k+l} f(x,y)g(x,y)}{\partial x^k\, \partial y^l} = \sum_{m=0}^{k} \sum_{n=0}^{l} \frac{k!}{m!(k-m)!} \frac{l!}{n!(l-n)!} \frac{\partial^{m+n} f(x,y)}{\partial x^m\, \partial y^n} \frac{\partial^{k+l-m-n} g(x,y)}{\partial x^{k-m}\, \partial y^{l-n}} \tag{307}$$

$$f(xy) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (xy)^k = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left. \frac{\partial^{2k} f(xy)}{\partial x^k\, \partial y^k} \right|_{x=y=0} (xy)^k \tag{308}$$

$$\frac{\partial (xy)^k}{\partial x\, \partial y} = k^2 (xy)^{k-1} = k \frac{d(xy)^k}{d(xy)} \tag{309}$$

$$\left. \frac{\partial^{2k} f(xy)}{\partial x^k\, \partial y^k} \right|_{x=y=0} = k! f^{(k)}(0) \tag{310}$$