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Subject: **Operator formalism and quasidistributions for creation and annihilation operators**
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1. Basics

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2}}(x+ip) & x &= \frac{1}{\sqrt{2}}(a+a^\dagger) & [a,a^\dagger] &= 1 \\
a^\dagger &= \frac{1}{\sqrt{2}}(x-ip) & p &= -\frac{i}{\sqrt{2}}(a-a^\dagger) & [x,p] &= i
\end{aligned} \tag{1}
$$

$$
\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \ge \frac{1}{4}\big|\langle[x,p]\rangle\big|^2 = \frac{1}{4} \tag{2}
$$

$$
\begin{aligned}
a^\dagger a &= \frac{1}{2}(x^2+p^2-1) & a^2 &= \frac{1}{2}\Big(x^2-p^2+i(xp+px)\Big) \\
aa^\dagger &= \frac{1}{2}(x^2+p^2+1) & (a^\dagger)^2 &= \frac{1}{2}\Big(x^2-p^2-i(xp+px)\Big)
\end{aligned} \tag{3}
$$

$$
[a,(a^\dagger)^n] = n(a^\dagger)^{n-1} \qquad [a^\dagger,a^n] = -na^{n-1} \tag{4}
$$

$$
[x,p^n] = inp^{n-1} \qquad [p,x^n] = -inx^{n-1} \tag{5}
$$

(Use $[A,B^n] = nB^{n-1}[A,B]$ if B commutes with $[A,B]$.)

$$
[a,a^\dagger a] = [a,a^\dagger]a = a \tag{6}
$$

$$
\begin{aligned}
e^{i\theta a^\dagger a}ae^{-i\theta a^\dagger a} &= ae^{-i\theta} \\
e^{i\theta a^\dagger a}a^\dagger e^{-i\theta a^\dagger a} &= a^\dagger e^{i\theta}
\end{aligned} \tag{7}
$$

(Use $e^ABe^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$)

$$
\begin{aligned}
e^{i\theta a^\dagger a}xe^{-i\theta a^\dagger a} &= x\cos\theta + p\sin\theta \\
e^{i\theta a^\dagger a}pe^{-i\theta a^\dagger a} &= -x\sin\theta + p\cos\theta
\end{aligned} \tag{8}
$$

2. Position and momentum bases

$$\begin{aligned} &\langle x'|x\rangle = \delta(x-x') && \langle p'|p\rangle = \delta(p-p') && \\ &\int dx\,|x\rangle\langle x| = 1 && \int dp\,|p\rangle\langle p| = 1 && \langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx} \end{aligned} \tag{9}$$

$$\begin{aligned} e^{-ipa}|x\rangle = |x+a\rangle \qquad \langle x|e^{-ipa} = \langle x-a| \\ e^{ixb}|p\rangle = |p+b\rangle \qquad \langle p|e^{ixb} = \langle p-b| \end{aligned} \tag{10}$$

$$\langle x|e^{-ipa}|\psi\rangle = \langle x-a|\psi\rangle \qquad \langle p|e^{ixb}|\psi\rangle = \langle p-b|\psi\rangle \tag{11}$$

$$\begin{aligned} \langle x|p|\psi\rangle &= \frac{1}{i}\frac{d}{da}\langle x|e^{ipa}|\psi\rangle\Big|_{a=0} = \frac{1}{i}\frac{d}{da}\langle x+a|\psi\rangle\Big|_{a=0} = \frac{1}{i}\frac{d}{dx}\langle x|\psi\rangle \iff p \leftrightarrow \frac{1}{i}\frac{d}{dx} \\ \langle p|x|\psi\rangle &= i\frac{d}{db}\langle p|e^{-ixb}|\psi\rangle\Big|_{b=0} = i\frac{d}{db}\langle p+b|\psi\rangle\Big|_{b=0} = i\frac{d}{dp}\langle p|\psi\rangle \iff x \leftrightarrow i\frac{d}{dp} \end{aligned} \tag{12}$$

$$a = \frac{1}{\sqrt{2}}(x+ip) \leftrightarrow \begin{cases} \frac{1}{\sqrt{2}}\left(x+\frac{d}{dx}\right) & \text{(position basis)} \\ \frac{i}{\sqrt{2}}\left(\frac{d}{dp}+p\right) & \text{(momentum basis)} \end{cases} \tag{13}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x-ip) \leftrightarrow \begin{cases} \frac{1}{\sqrt{2}}\left(x-\frac{d}{dx}\right) & \text{(position basis)} \\ \frac{i}{\sqrt{2}}\left(\frac{d}{dp}-p\right) & \text{(momentum basis)} \end{cases} \tag{14}$$

3. Displacement operator

$$D(a,\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} = e^{i(\alpha_2 x - \alpha_1 p)} = e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}e^{-i\alpha_1 p} = e^{i\alpha_1\alpha_2/2}e^{-i\alpha_1 p}e^{i\alpha_2 x} \tag{15}$$

$$\alpha = \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) \qquad \begin{aligned} \alpha_1 &= \frac{1}{\sqrt{2}}(\alpha+\alpha^*) \\ \alpha_2 &= -\frac{i}{\sqrt{2}}(\alpha-\alpha^*) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_1} - i\frac{\partial}{\partial\alpha_2}\right) & \frac{\partial}{\partial\alpha_1} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right) \\ \frac{\partial}{\partial\alpha^*} &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_1} + i\frac{\partial}{\partial\alpha_2}\right) & \frac{\partial}{\partial\alpha_2} &= \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\alpha^*}\right) \end{aligned} \tag{17}$$

$$\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\alpha_2^2}\right) \tag{18}$$

$$D^{-1}(a,\alpha) = D^\dagger(a,\alpha) = D(a,-\alpha) = D(-a,\alpha) \tag{19}$$

$$
\begin{aligned}
&D(a,\alpha)|x\rangle = e^{i\alpha_1\alpha_2/2}e^{i\alpha_2 x}|x+\alpha_1\rangle \qquad \langle x|D(a,\alpha) = e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}\langle x-\alpha_1| \\
&D(a,\alpha)|p\rangle = e^{-i\alpha_1\alpha_2/2}e^{-i\alpha_1 p}|p+\alpha_2\rangle \qquad \langle p|D(a,\alpha) = e^{i\alpha_1\alpha_2/2}e^{-i\alpha_1 p}\langle p-\alpha_2|
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\langle x|D(a,\alpha)|x'\rangle &= e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}\delta(x-x'-\alpha_1) \\
\langle p|D(a,\alpha)|p'\rangle &= e^{i\alpha_1\alpha_2/2}e^{-i\alpha_1 p}\delta(p-p'-\alpha_2)
\end{aligned} \tag{21}
$$

$$D(\alpha,\beta) = e^{\beta\alpha^*-\beta^*\alpha} = e^{2i(\beta_I\alpha_R-\beta_R\alpha_I)} = e^{i(\beta_2\alpha_1-\beta_1\alpha_2)} \tag{22}$$

$$D(\alpha,\alpha) = 1 \qquad D(\alpha,r\alpha) = 1,\ r \text{ real} \tag{23}$$

$$D^*(\alpha,\beta) = D(\alpha^*,\beta^*) = D(\alpha,-\beta) = D(-\alpha,\beta) = D(\beta,\alpha) \tag{24}$$

$$D(a,\alpha) = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a} = e^{|\alpha|^2/2}e^{-\alpha^* a}e^{\alpha a^\dagger} \tag{25}$$

(Use BCH: $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if A and B commute with $[A,B]$.)

$$D^\dagger(a,\alpha)aD(a,\alpha) = a+\alpha \tag{26}$$

$$D^\dagger(a,\alpha)D(a,\beta)D(a,\alpha) = D(a+\alpha,\beta) = D(\alpha,\beta)D(a,\alpha) \tag{27}$$

$$
\begin{aligned}
D(a,\alpha)D(a,\beta) &= e^{(\alpha\beta^*-\alpha^*\beta)/2}D(a,\alpha+\beta) \\
&= D(\beta,\alpha/2)D(a,\alpha+\beta) \\
&= D(\beta,\alpha)D(a,\beta)D(a,\alpha)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&D(a,\alpha)D(a,\beta) = D(a,\beta)D(a,\alpha) \quad \Longleftrightarrow \quad D(\beta,\alpha) = 1 \\
&\Longleftrightarrow \quad \begin{pmatrix}\text{area subtended by} \\ \alpha \text{ and } \beta\end{pmatrix} = \frac{1}{2i}(\alpha\beta^*-\alpha^*\beta) = \alpha_I\beta_R-\alpha_R\beta_I = \pi k \\
&\qquad\quad \begin{pmatrix}\text{area subtended by} \\ (\alpha_1,\alpha_2) \text{ and } (\beta_1,\beta_2)\end{pmatrix} = \alpha_2\beta_1-\alpha_1\beta_2 = 2\pi k
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
D(a,\alpha)D^\dagger(a,\beta) &= e^{-(\alpha\beta^*-\alpha^*\beta)/2}D(a,\alpha-\beta) \\
&= D(\beta,-\alpha/2)D(a,\alpha-\beta) \\
&= D(\beta,-\alpha)D^\dagger(a,\beta)D(a,\alpha)
\end{aligned} \tag{30}
$$

$$D^\dagger(a,\beta)D(a,\alpha) = e^{(\alpha\beta^*-\alpha^*\beta)/2}D(a,\alpha-\beta) = D(\beta,\alpha/2)D(a,\alpha-\beta) \tag{31}$$

$$e^{i\theta a^\dagger a}D(a,\alpha)e^{-i\theta a^\dagger a} = D(ae^{-i\theta},\alpha) = D(a,\alpha e^{i\theta}) \tag{32}$$

4. Number states

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{33}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad\qquad a|n\rangle = \sqrt{n}|n-1\rangle \tag{34}$$

$$a^\dagger a|n\rangle = n|n\rangle \tag{35}$$

$$\langle n|m\rangle = \delta_{nm} \tag{36}$$

$$\langle n|a|n\rangle = 0 \qquad \begin{matrix} \langle n|a^2|n\rangle = 0 \\ \langle n|a^\dagger a|n\rangle = n \end{matrix} \tag{37}$$

$$\begin{matrix} \langle n|x|n\rangle = 0 \\ \langle n|p|n\rangle = 0 \end{matrix} \qquad \begin{matrix} \langle n|x^2|n\rangle = \langle n|p^2|n\rangle = n + \frac{1}{2} \\ \langle n|(xp+px)|n\rangle = 0 \end{matrix} \tag{38}$$

$$\begin{aligned} e^{-\alpha^* a}|n\rangle &= \sum_{k=0}^{\infty}\frac{(-\alpha^*)^k}{k!}a^k|n\rangle = \sum_{k=0}^{n}\frac{(-\alpha^*)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle \\ \langle n|e^{\alpha a^\dagger} &= \sum_{k=0}^{n}\frac{\alpha^k}{k!}\sqrt{\frac{n!}{(n-k)!}}\langle n-k| \end{aligned} \tag{39}$$

$$\begin{aligned} m \ge n: \quad \langle m|e^{\alpha a^\dagger}e^{-\alpha^* a}|n\rangle &= \sum_{l=0}^{m}\sum_{k=0}^{n}\frac{\alpha^l(-\alpha^*)^k}{l!\,k!}\sqrt{\frac{m!\,n!}{(m-l)!\,(n-k)!}}\langle m-l|n-k\rangle \\ &= \sum_{k=0}^{n}\frac{\alpha^{m-n}(-|\alpha|^2)^k}{(m-n+k)!\,k!}\frac{\sqrt{m!\,n!}}{(n-k)!} \qquad \text{(Here we use } m \ge n.) \\ &= \sqrt{\frac{n!}{m!}}\,\alpha^{m-n}\sum_{k=0}^{n}\frac{(n+m-n)!}{k!\,(n-k)!\,(m-n+k)!}(-|\alpha|^2)^k \\ &= \sqrt{\frac{n!}{m!}}\,\alpha^{m-n}L_n^{(m-n)}\big(|\alpha|^2\big) \end{aligned} \tag{40}$$

[$L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial of A&S (22.3.9).]

$$\langle m|D(a,\alpha)|n\rangle = \begin{cases} \sqrt{\dfrac{n!}{m!}}\,e^{-|\alpha|^2/2}\alpha^{m-n}L_n^{(m-n)}\big(|\alpha|^2\big)\,, & m \ge n \\ \sqrt{\dfrac{m!}{n!}}\,e^{-|\alpha|^2/2}(-\alpha^*)^{n-m}L_m^{(n-m)}\big(|\alpha|^2\big)\,, & m \le n \end{cases} \tag{41}$$

$$\langle m|D(a,\alpha^*)|n\rangle = (-1)^{m-n}\langle n|D(a,\alpha)|m\rangle \tag{42}$$

$$\langle n|D(a,\alpha)|n\rangle = e^{-|\alpha|^2/2}L_n\big(|\alpha|^2\big) \tag{43}$$

$$e^{-i\theta a^\dagger a}|n\rangle = e^{-in\theta}|n\rangle \tag{44}$$

$$0 = \langle x|a|0\rangle = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\langle x|0\rangle \qquad \Longrightarrow \qquad \langle x|0\rangle = \frac{1}{\pi^{1/4}}e^{-x^2/2}$$

(phase chosen by convention) (45)

$$\langle p|0\rangle = \int_{-\infty}^{\infty} dx\,\langle p|x\rangle\langle x|0\rangle = \frac{1}{\pi^{1/4}}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,e^{-ipx}e^{-x^2/2} = \frac{1}{\pi^{1/4}}e^{-p^2/2}$$

$$\begin{aligned}\langle x|n\rangle &= \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|0\rangle \\ &= \frac{1}{\sqrt{2^n n!}}\langle x|(x-ip)^n|0\rangle \\ &= \frac{1}{\sqrt{2^n n!}}\left(x - \frac{d}{dx}\right)^n\langle x|0\rangle \\ &= \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\underbrace{\left(x - \frac{d}{dx}\right)^n e^{-x^2/2}}_{=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}} \\ &= \frac{e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}}\underbrace{(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}}_{=H_n(x)}\end{aligned} \tag{46}$$

[Use Rodrigues's formula for the Hermite polynomial $H_n(x)$: A&S (22.11.7).]

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}e^{-x^2/2}H_n(x) = \frac{1}{\sqrt{2^n n!}}H_n(x)\langle x|0\rangle \tag{47}$$

5. Normal ordering

Normal ordering, denoted by paired colons, applies to functions of creation and annihilation operators, i.e., expressions written in terms of a and $a^\dagger$. It means to move all annihilation operators to the right and all creation operators to the left without regard to commutators. It is meaningless to refer to the normal-ordered form of an operator A, i.e., to write $:\!A\!:$, because the result of normal ordering depends on how A is written in terms of creation and annihilation operators. For example, if $A = aa^\dagger = a^\dagger a + 1$, the result of normal ordering the first form is $a^\dagger a$, but the result of normal ordering the second form is $a^\dagger a + 1$.

$$\begin{aligned}:\!(a^\dagger a)^k\!: &\equiv (a^\dagger)^k a^k \\ &= \sum_{n,m=0}^{\infty}|n\rangle\langle n|(a^\dagger)^k a^k|m\rangle\langle m| \\ &= \sum_{n=k}^{\infty}|n\rangle\langle n|(a^\dagger)^k a^k|n\rangle\langle n| \\ &= \sum_{n=k}^{\infty}\frac{n!}{(n-k)!}|n\rangle\langle n|\end{aligned} \tag{48}$$

$$
\begin{aligned}
{:}f(a^\dagger a){:} &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} {:}(a^\dagger a)^k{:} \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} |n\rangle\langle n| \qquad (49) \\
&= \sum_{n=0}^{\infty} |n\rangle\langle n| \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} f^{(k)}(0)
\end{aligned}
$$

$$
{:}e^{-\lambda a^\dagger a}{:} = \sum_{n=0}^{\infty} |n\rangle\langle n| \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}(-\lambda)^k = \sum_{n=0}^{\infty} |n\rangle\langle n|(1-\lambda)^n = (1-\lambda)^{a^\dagger a} = e^{\ln(1-\lambda) a^\dagger a} \qquad (50)
$$

$$
\begin{aligned}
{:}e^{-\lambda(a^\dagger-\alpha^*)(a-\alpha)}{:} &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger - \alpha^*)^k (a-\alpha)^k \\
&= D(a,\alpha) \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} (a^\dagger)^k a^k \right) D^\dagger(a,\alpha) \qquad (51) \\
&= D(a,\alpha) {:}e^{-\lambda a^\dagger a}{:} D^\dagger(a,\alpha) \\
&= (1-\lambda)^{(a^\dagger-\alpha^*)(a-\alpha)}
\end{aligned}
$$

6. Coherent states

$$
|\alpha\rangle \equiv D(a,\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad (52)
$$

$$
\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \qquad (53)
$$

$$
a|\alpha\rangle = \alpha|\alpha\rangle \qquad \langle\alpha|a^\dagger = \langle\alpha|\alpha^* \qquad (54)
$$

$$
e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-i\theta a^\dagger a} D(a,\alpha)|0\rangle = D(a,\alpha e^{-i\theta}) e^{-i\theta a^\dagger a}|0\rangle = |\alpha e^{-i\theta}\rangle \qquad (55)
$$

$$
\langle\alpha|a|\alpha\rangle = \alpha \qquad \begin{aligned} \langle\alpha|a^2|\alpha\rangle &= \alpha^2 \\ \langle\alpha|a^\dagger a|\alpha\rangle &= |\alpha|^2 \end{aligned} \qquad (56)
$$

$$
\begin{aligned} \langle\alpha|x|\alpha\rangle &= \alpha_1 \\ \langle\alpha|p|\alpha\rangle &= \alpha_2 \end{aligned} \qquad \begin{aligned} \langle\alpha|(\Delta x)^2|\alpha\rangle = \langle\alpha|(\Delta p)^2|\alpha\rangle &= \frac{1}{2} \\ \langle\alpha|(\Delta x \Delta p + \Delta p \Delta x)|\alpha\rangle &= 0 \end{aligned} \qquad (57)
$$

$$
\begin{aligned}
\langle x|\alpha\rangle &= \langle x|D(a,\alpha)|0\rangle = e^{-i\alpha_1\alpha_2/2} e^{i\alpha_2 x} \langle x-\alpha_1|0\rangle = \frac{e^{-i\alpha_1\alpha_2/2}}{\pi^{1/4}} e^{-(x-\alpha_1)^2/2} e^{i\alpha_2 x} \\
\langle p|\alpha\rangle &= \langle p|D(a,\alpha)|0\rangle = e^{i\alpha_1\alpha_2/2} e^{-i\alpha_1 p} \langle p-\alpha_2|0\rangle = \frac{e^{i\alpha_1\alpha_2/2}}{\pi^{1/4}} e^{-(p-\alpha_2)^2/2} e^{-i\alpha_1 p}
\end{aligned} \qquad (58)
$$

$$
\langle x|\alpha\rangle = \langle -x|-\alpha\rangle \qquad \langle p|\alpha\rangle = \langle -p|-\alpha\rangle \qquad (59)
$$

$$\langle 0|D(a,\alpha)|0\rangle = \langle 0|\alpha\rangle = e^{-|\alpha|^2/2} \tag{60}$$

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(a,\beta)D(a,\alpha)|0\rangle = D(\beta,\alpha/2)e^{-|\alpha-\beta|^2/2} = e^{-|\alpha|^2/2}e^{-|\beta|^2/2}e^{\alpha\beta^*} \tag{61}$$

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{62}$$

$$\begin{aligned} D(a,\alpha)|\beta\rangle &= D(a,\alpha)D(a,\beta)|0\rangle = D(\beta,\alpha/2)|\beta+\alpha\rangle \\ \langle\beta|D(a,\alpha) &= \langle 0|D^\dagger(a,\beta)D(a,\alpha) = \langle\beta-\alpha|D(\beta,\alpha/2) \end{aligned} \tag{63}$$

$$\langle\beta|D(a,\alpha)|\beta\rangle = e^{-|\alpha|^2/2}\langle\beta|e^{\alpha a^\dagger}e^{-\alpha^* a}|\beta\rangle = e^{-|\alpha|^2/2}D(\beta,\alpha) \tag{64}$$

$$\begin{aligned} \langle\gamma|D(a,\alpha)|\beta\rangle &= e^{-|\alpha|^2/2}\langle\gamma|e^{\alpha a^\dagger}e^{-\alpha^* a}|\beta\rangle \\ &= e^{-|\alpha|^2/2}e^{\alpha\gamma^*-\alpha^*\beta}\langle\gamma|\beta\rangle \\ &= e^{-|\alpha|^2/2}e^{-|\beta|^2/2}e^{-|\gamma|^2/2}e^{\alpha\gamma^*-\alpha^*\beta+\beta\gamma^*} \end{aligned} \tag{65}$$

$$\langle\gamma|D(a,\alpha)|\beta\rangle = D(\gamma,\alpha/2)D(\beta,\alpha/2)D(\gamma,\beta/2)e^{-|\alpha+\beta-\gamma|^2/2} \tag{66}$$

$$e^{-\lambda a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-\lambda})^n}{\sqrt{n!}}|n\rangle = e^{-|\alpha|^2(1-e^{-2\lambda})/2}|\alpha e^{-\lambda}\rangle \tag{67}$$

$$\langle\alpha|e^{-i\theta a^\dagger a}|\alpha\rangle = e^{-|\alpha|^2(1-e^{-i\theta})} \tag{68}$$

$$\langle\alpha|e^{-\lambda a^\dagger a}|\alpha\rangle = \langle\alpha|{:}e^{-(1-e^{-\lambda})a^\dagger a}{:}|\alpha\rangle = e^{-(1-e^{-\lambda})|\alpha|^2} \tag{69}$$

$$\begin{aligned} \alpha &= \alpha_R + i\alpha_I = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) = |\alpha|e^{i\phi} \\ d^2\alpha &= d\alpha_R\,d\alpha_I = \frac{d\alpha_1\,d\alpha_2}{2} = |\alpha|d|\alpha|\,d\phi = \frac{1}{2}d|\alpha|^2\,d\phi \end{aligned} \tag{70}$$

$$\begin{aligned} \left\langle n\middle|\left(\int d^2\alpha\,|\alpha\rangle\langle\alpha|\right)\middle|m\right\rangle &= \int d^2\alpha\,\langle n|\alpha\rangle\langle\alpha|m\rangle \\ &= \frac{1}{\sqrt{n!\,m!}}\int d^2\alpha\,e^{-|\alpha|^2}\alpha^n(\alpha^*)^m \\ &= \frac{1}{2\sqrt{n!\,m!}}\int d|\alpha|^2\,d\phi\,e^{-|\alpha|^2}|\alpha|^{n+m}e^{i(n-m)\phi} \\ &= \frac{\pi}{n!}\delta_{nm}\int_0^\infty du\,e^{-u}u^n \\ &= \pi\delta_{nm} \end{aligned} \tag{71}$$

$$1 = \int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)|0\rangle\langle 0|D^\dagger(a,\alpha) \tag{72}$$

$$\mathrm{tr}(A) = \int\frac{d^2\alpha}{\pi}\,\langle\alpha|A|\alpha\rangle = \int\frac{d^2\alpha}{\pi}\,\langle 0|D^\dagger(a,\alpha)AD(a,\alpha)|0\rangle \tag{73}$$

$$\begin{aligned} \mathcal{I} = 1\odot 1 &= \sum_{n,m}|n\rangle\langle n|\odot|m\rangle\langle m| = \int\frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,|\alpha\rangle\langle\alpha|\odot|\beta\rangle\langle\beta| \\ \mathbf{I} = \mathcal{I}^\# &= \sum_{n,m}|n\rangle\langle m|\odot|m\rangle\langle n| = \int\frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,|\alpha\rangle\langle\beta|\odot|\beta\rangle\langle\alpha| \end{aligned} \tag{74}$$

7. Parity

$$\begin{aligned} P^\dagger x P &= -x \\ P^\dagger p P &= -p \end{aligned} \qquad \Longleftrightarrow \qquad P^\dagger a P = -a \tag{75}$$

$$aP|0\rangle = -Pa|0\rangle = 0 \quad \Longrightarrow \quad P|0\rangle = e^{i\delta}|0\rangle = |0\rangle \quad \text{(choose phase } \delta = 0\text{)} \tag{76}$$

$$P|n\rangle = \frac{1}{\sqrt{n!}} P(a^\dagger)^n|0\rangle = \frac{(-1)^n}{\sqrt{n!}}(a^\dagger)^n P|0\rangle = (-1)^n|n\rangle \tag{77}$$

$$P = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n| = (-1)^{a^\dagger a} = P^\dagger \tag{78}$$

$$PD(a,\alpha)P = D(a,-\alpha) = D^\dagger(a,\alpha) \tag{79}$$

$$P|\alpha\rangle = PD(a,\alpha)|0\rangle = D(a,-\alpha)P|0\rangle = D(a,-\alpha)|0\rangle = |-\alpha\rangle \tag{80}$$

$$P = \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle\alpha| \tag{81}$$

$$\begin{aligned} P|x\rangle &= \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha|x\rangle = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle-\alpha|-x\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|-x\rangle = |-x\rangle \\ P|p\rangle &= \int \frac{d^2\alpha}{\pi}\, P|\alpha\rangle\langle\alpha|p\rangle = \int \frac{d^2\alpha}{\pi}\, |-\alpha\rangle\langle-\alpha|-p\rangle = \int \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha|-p\rangle = |-p\rangle \end{aligned} \tag{82}$$

$$\begin{aligned} P &= \int dx\, P|x\rangle\langle x| = \int dx\, |-x\rangle\langle x| \\ P &= \int dp\, P|p\rangle\langle p| = \int dp\, |-p\rangle\langle p| \end{aligned} \tag{83}$$

$$\begin{aligned} \operatorname{tr}\big(PD(a,\alpha)\big) &= \int \frac{d^2\beta}{\pi}\, \langle\beta|PD(a,\alpha)|\beta\rangle \\ &= \int \frac{d^2\beta}{\pi}\, \langle-\beta|D(a,\alpha)|\beta\rangle \\ &= e^{-|\alpha|^2/2}\int \frac{d^2\beta}{\pi}\, e^{-\alpha\beta^*-\alpha^*\beta}\langle-\beta|\beta\rangle \\ &= e^{-|\alpha|^2/2}\underbrace{\int \frac{d^2\beta}{\pi}\, e^{-2|\beta|^2}e^{-\beta\alpha^*-\beta^*\alpha}}_{=\frac{1}{2}e^{|\alpha|^2/2}} \\ &= \frac{1}{2} \end{aligned} \tag{84}$$

8. Fourier transform pairs

$$\int \frac{d^2\beta}{\pi}\, D(\beta,\alpha) = \int \frac{d^2\beta}{\pi}\, e^{\alpha\beta^* - \alpha^*\beta} = \int \frac{d\beta_1\, d\beta_2}{2\pi}\, e^{i(\alpha_2\beta_1 - \alpha_1\beta_2)} = 2\pi\delta(\alpha_2)\delta(\alpha_1) = \pi\delta(\alpha) \tag{85}$$

$$f(\alpha) = \int \frac{d^2\beta}{\pi}\, \tilde{f}(\beta) D(\beta,\alpha) \qquad\qquad \tilde{f}(\beta) = \int \frac{d^2\alpha}{\pi}\, f(\alpha) D(\alpha,\beta) \tag{86}$$

$$g(\alpha) = f^*(\alpha) \qquad \Longleftrightarrow \qquad \tilde{g}(\beta) = \tilde{f}^*(-\beta) \tag{87}$$

$$\begin{aligned} \frac{\partial\tilde{f}}{\partial\beta} &= \int \frac{d^2\alpha}{\pi}\, \alpha^* f(\alpha) D(\alpha,\beta) \\ \frac{\partial\tilde{f}}{\partial\beta^*} &= -\int \frac{d^2\alpha}{\pi}\, \alpha f(\alpha) D(\alpha,\beta) \end{aligned} \qquad \frac{\partial^2\tilde{f}}{\partial\beta\,\partial\beta^*} = -\int \frac{d^2\alpha}{\pi}\, |\alpha|^2 f(\alpha) D(\alpha,\beta) \tag{88}$$

$$\begin{aligned} &\int \frac{d^2\alpha}{\pi}\, f(\alpha) g(\alpha) D(\alpha,\beta) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(\beta-\gamma) \\ &\int \frac{d^2\beta}{\pi} \left(\int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(\beta-\gamma) \right) D(\beta,\alpha) = f(\alpha) g(\alpha) \end{aligned} \tag{89}$$

$$\int \frac{d^2\alpha}{\pi}\, f(\alpha) g(\alpha) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{g}(-\gamma) \tag{90}$$

$$\int \frac{d^2\alpha}{\pi}\, |f(\alpha)|^2 D(\alpha,\beta) = \int \frac{d^2\gamma}{\pi}\, \tilde{f}(\gamma)\tilde{f}^*(\gamma-\beta) \qquad \text{(Parseval's relation)} \tag{91}$$

$$\int \frac{d^2\alpha}{\pi}\, |f(\alpha)|^2 = \int \frac{d^2\beta}{\pi}\, |\tilde{f}(\beta)|^2 \tag{92}$$

9. Gaussian integrals

$$\int_{-\infty}^{\infty} du\, e^{-ax^2} e^{bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a} \tag{93}$$

$$\int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} D(\alpha,\beta) = \int \frac{d\alpha_1\, d\alpha_2}{2\pi}\, e^{-(\alpha_1^2+\alpha_2^2)/2} e^{i(\beta_2\alpha_1 - \beta_1\alpha_2)} = e^{-(\beta_1^2+\beta_2^2)/2} = e^{-|\beta|^2} \tag{94}$$

$$\int \frac{d^2\alpha}{\pi\sigma^2}\, e^{-|\alpha|^2/\sigma^2} D(\alpha,\beta) = \int \frac{d^2\alpha'}{\pi}\, e^{-|\alpha'|^2} D(\alpha',\sigma\beta) = e^{-\sigma^2|\beta|^2} \tag{95}$$

$$\begin{aligned} \int \frac{d^2\alpha}{\pi}\, e^{-|\alpha|^2} e^{\alpha\gamma^* - \alpha^*\beta} &= \int \frac{d\alpha_1\, d\alpha_2}{2\pi}\, e^{-(\alpha_1^2+\alpha_2^2)/2} e^{\alpha_1(\gamma^*-\beta)/\sqrt{2}} e^{i\alpha_2(\gamma^*+\beta)/\sqrt{2}} \\ &= e^{(\gamma^*-\beta)^2/4} e^{-(\gamma^*+\beta)^2/4} \\ &= e^{-\beta\gamma^*} \end{aligned} \tag{96}$$

$$\int \frac{d^2\alpha}{\pi\sigma^2}\, e^{-|\alpha|^2/\sigma^2} e^{\alpha\gamma^* - \alpha^*\beta} = \int \frac{d^2\alpha'}{\pi}\, e^{-|\alpha'|^2} e^{\alpha'\sigma\gamma^* - \alpha'^*\sigma\beta} = e^{-\sigma^2\beta\gamma^*} \tag{97}$$

10. Orthogonality and completeness of displacement operators

$$\mathrm{tr}\big(D(a,\alpha)\big) = \int \frac{d^2\beta}{\pi}\,\langle\beta|D(a,\alpha)|\beta\rangle = e^{-|\alpha|^2/2}\int \frac{d^2\beta}{\pi}\,D(\beta,\alpha) = \pi\delta(\alpha) \tag{98}$$

$$\mathrm{tr}\big(D^\dagger(a,\beta)D(a,\alpha)\big) = D(\beta,\alpha/2)\mathrm{tr}\big(D(a,\alpha-\beta)\big) = \pi\delta(\alpha-\beta) \tag{99}$$

$$\begin{aligned}\int \frac{d^2\alpha}{\pi}\,\langle\mu|D(a,\alpha)|\beta\rangle\langle\gamma|D^\dagger(a,\alpha)|\nu\rangle &= \langle\mu|\beta\rangle\langle\gamma|\nu\rangle\int\frac{d^2\alpha}{\pi}\,e^{-|\alpha|^2}e^{\alpha(\mu^*-\gamma^*)-\alpha^*(\beta-\nu)}\\ &= \langle\mu|\beta\rangle\langle\gamma|\nu\rangle e^{-(\beta-\nu)(\mu^*-\gamma^*)}\\ &= e^{-|\mu|^2/2}e^{-|\beta|^2/2}e^{\beta\mu^*}\\ &\quad\times e^{-|\gamma|^2/2}e^{-|\nu|^2/2}e^{\nu\gamma^*}e^{-(\beta-\nu)(\mu^*-\gamma^*)}\\ &= e^{-|\beta|^2/2}e^{-|\gamma|^2/2}e^{\beta\gamma^*}e^{-|\nu|^2/2}e^{-|\mu|^2/2}e^{\nu\mu^*}\\ &= \langle\gamma|\beta\rangle\langle\mu|\nu\rangle\end{aligned} \tag{100}$$

$$\begin{aligned}\mathrm{tr}\left((a^\dagger)^k a^l\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\beta)\right) &= \int\frac{d^2\alpha}{\pi}\,(\alpha^*)^k\alpha^l D(\alpha,\beta)\\ &= \frac{\partial^{k+l}}{\partial\beta^k\partial(-\beta^*)^l}\underbrace{\int\frac{d^2\alpha}{\pi}\,D(\alpha,\beta)}_{=\pi\delta(\beta)=\mathrm{tr}\left(D(a,\beta)\right)}\\ &= \frac{\partial^{k+l}}{\partial\beta^k\partial(-\beta^*)^l}\mathrm{tr}\big(e^{-\beta^* a}e^{\beta a^\dagger}\big)\\ &= \mathrm{tr}\big((a^\dagger)^k a^l e^{-\beta^* a}e^{\beta a^\dagger}\big)\end{aligned} \tag{101}$$

$$\begin{aligned}&\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\beta) = e^{-\beta^* a}e^{\beta a^\dagger} = e^{-|\beta|^2/2}D(a,\beta)\\ &\int\frac{d^2\beta}{\pi}\,e^{-|\beta|^2/2}D(a,\beta)D(\beta,\alpha) = |\alpha\rangle\langle\alpha|\end{aligned} \tag{102}$$

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}\,D(a,\alpha)|\beta\rangle\langle\gamma|D^\dagger(a,\alpha) &= \int\frac{d^2\alpha}{\pi}\,D(a,\alpha)D(a,\beta)|0\rangle\langle 0|D^\dagger(a,\gamma)D^\dagger(a,\alpha)\\ &= D(a,\beta)\left(\int\frac{d^2\alpha}{\pi}\,D(a+\beta,\alpha)|0\rangle\langle 0|D^\dagger(a+\gamma,\alpha)\right)D^\dagger(a,\gamma)\\ &= D(a,\beta)\left(\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|D(\alpha,\gamma-\beta)\right)D^\dagger(a,\gamma)\\ &= e^{-|\gamma-\beta|^2/2}D(a,\beta)D(a,\gamma-\beta)D^\dagger(a,\gamma)\\ &= D(\gamma,\beta/2)e^{-|\beta-\gamma|^2/2}D(a,\beta)D^\dagger(a,\beta)D(a,\gamma)D^\dagger(a,\gamma)\\ &= \langle\gamma|\beta\rangle\,1\end{aligned} \tag{103}$$

$$\mathbf{I} = \int \frac{d^2\alpha}{\pi}\, D(a,\alpha) \odot D^\dagger(a,\alpha) = \int \frac{d^2\alpha}{\pi}\, D^\dagger(a,\alpha) \odot D(a,\alpha) \tag{104}$$

[Follows from Eq. (100) or Eq. (103)]

$$\mathrm{tr}(A)1 = \mathbf{I}(A) = \int \frac{d^2\alpha}{\pi}\, D(a,\alpha) A D^\dagger(a,\alpha) \tag{105}$$

$$1 = \int \frac{d^2\alpha}{\pi}\, D(a,\alpha)\rho D^\dagger(a,\alpha) = \int \frac{d^2\alpha}{\pi}\, D(a,\alpha)|\psi\rangle\langle\psi| D^\dagger(a,\alpha) \tag{106}$$

$$\mathrm{tr}(A) = \int \frac{d^2\alpha}{\pi}\, \mathrm{tr}\big(D(a,\alpha)\rho D^\dagger(a,\alpha)A\big) = \int \frac{d^2\alpha}{\pi}\, \langle\psi| D^\dagger(a,\alpha) A D(a,\alpha)|\psi\rangle \tag{107}$$

11. Operator ordering

$$D^{(s)}(a,\alpha) \equiv e^{s|\alpha|^2/2} D(a,\alpha) \tag{108}$$

$$D^{(s)\dagger}(a,\alpha) = D^{(s)}(-a,\alpha) = D^{(s)}(a,-\alpha) \tag{109}$$

$$\mathrm{tr}\big(D^{(s)}(a,\alpha)\big) = \pi\delta(\alpha) \tag{110}$$

$$\mathrm{tr}\big(D^{(-s)\dagger}(a,\beta) D^{(s)}(a,\alpha)\big) = \pi\delta(\alpha-\beta) \tag{111}$$

$$\mathbf{I} = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha) \odot D^{(s)}(a,\alpha) \tag{112}$$

$$\begin{aligned} s = +1 \text{ (normal ordering):} \quad & D^{(+1)}(a,\alpha) = e^{\alpha a^\dagger} e^{-\alpha^* a} = \mathbf{:}D(a,\alpha)\mathbf{:} \\ s = 0 \text{ (symmetric ordering):} \quad & D^{(0)}(a,\alpha) = D(a,\alpha) = e^{\alpha a^\dagger - \alpha^* a} \\ s = -1 \text{ (antinormal ordering):} \quad & D^{(-1)}(a,\alpha) = e^{-\alpha^* a} e^{\alpha a^\dagger} \end{aligned} \tag{113}$$

$$\begin{aligned} [(a^\dagger)^k a^l]_{(s)} &\equiv \left. \frac{\partial^{k+l} D^{(s)}(a,\alpha)}{\partial\alpha^k \partial(-\alpha^*)^l} \right|_{\alpha=0} \\ D^{(s)}(a,\alpha) &= \sum_{k,l} \frac{\alpha^k (-\alpha^*)^l}{k!\, l!} [(a^\dagger)^k a^l]_{(s)} \end{aligned} \tag{114}$$

$$\begin{aligned} [(a^\dagger)^k a^l]_{(+1)} &= (a^\dagger)^k a^l \\ [(a^\dagger)^k a^l]_{(-1)} &= a^l (a^\dagger)^k \end{aligned} \tag{115}$$

$$\begin{aligned} \tilde{D}^{(s)}(a,\beta) &\equiv \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a,\alpha) D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a-\beta,\alpha) \equiv \delta^{(s)}(a-\beta) \\ D^{(s)}(a,\alpha) &= \int \frac{d^2\beta}{\pi}\, \tilde{D}^{(s)}(a,\beta) D(\beta,\alpha) \end{aligned} \tag{116}$$

$$\tilde{D}^{(s)\dagger}(a,\beta) = \tilde{D}^{(s)}(a,\beta) \tag{117}$$

$$\mathrm{tr}\big(\tilde{D}^{(s)}(a,\alpha)\big) = 1 \tag{118}$$

$$\int d^2\beta\, \tilde{D}^{(s)}(a,\beta) = D^{(s)}(a,0) = 1 \tag{119}$$

$$\begin{aligned}
\operatorname{tr}\big(\tilde{D}^{(-s)\dagger}(a,\gamma)\tilde{D}^{(s)}(a,\beta)\big) &= \int \frac{d^2\alpha'}{\pi}\,\frac{d^2\alpha}{\pi}\,\underbrace{\operatorname{tr}\big(D^{(-s)\dagger}(a,\alpha')D^{(s)}(a,\alpha)\big)}_{=\pi\delta(\alpha-\alpha')} D(\alpha',-\gamma)D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, D(\alpha,\beta-\gamma) \\
&= \pi\delta(\beta-\gamma)
\end{aligned} \tag{120}$$

$$\begin{aligned}
\mathbf{I} &= \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha)\odot D^{(s)}(a,\alpha) \\
&= \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha)\odot \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)D(\beta,\alpha) \\
&= \int \frac{d^2\beta}{\pi}\,\left(\int \frac{d^2\alpha}{\pi}\, D^{(-s)}(a,\alpha)D(\alpha,\beta)\right)^{\dagger}\odot \tilde{D}^{(s)}(a,\beta) \\
&= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(-s)}(a,\beta)\odot\tilde{D}^{(s)}(a,\beta)
\end{aligned} \tag{121}$$

$$\begin{aligned}
\tilde{D}^{(-s)\dagger}(a,\gamma)\tilde{D}^{(s)}(a,\beta) &= \int \frac{d^2\alpha'}{\pi}\,\frac{d^2\alpha}{\pi}\, D(\alpha',-\gamma)D(\alpha,\beta)\underbrace{D^{(-s)\dagger}(a,\alpha')D^{(s)}(a,\alpha)}_{=D(\alpha',\alpha/2)D(a,\alpha-\alpha')} \\
&= \int \frac{d^2\mu}{\pi}\, D(a,\mu)\int \frac{d^2\nu}{\pi}\, D(\nu,\mu/2)D(\nu-\mu/2,-\gamma)D(\nu+\mu/2,\beta) \\
&= \int \frac{d^2\mu}{\pi}\, D(a,\mu)D\big(\mu,(\beta+\gamma)/2\big)\underbrace{\int \frac{d^2\nu}{\pi}\, D(\nu,\mu/2+\beta-\gamma)}_{=4\pi\delta\big(\mu-2(\gamma-\beta)\big)} \\
&= 4D\big(a,2(\gamma-\beta)\big)D(\gamma-\beta,\beta+\gamma) \\
&= 4D\big(a,2(\gamma-\beta)\big)D(\gamma,2\beta) \\
&= 4D^{\dagger}(a,-2\gamma)D(a,-2\beta)D(\gamma,2\beta)
\end{aligned} \tag{122}$$

$$\begin{aligned}
[(a^{\dagger})^k a^l]_{(s)} &= \left.\frac{\partial^{k+l}D^{(s)}(a,\alpha)}{\partial\alpha^k\partial(-\alpha^*)^l}\right|_{\alpha=0} \\
&= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)\,\underbrace{\left.\frac{\partial^{k+l}D(\beta,\alpha)}{\partial\alpha^k\partial(-\alpha^*)^l}\right|_{\alpha=0}}_{=(\beta^*)^k\beta^l} \\
&= \int \frac{d^2\beta}{\pi}\,\tilde{D}^{(s)}(a,\beta)(\beta^*)^k\beta^l
\end{aligned} \tag{123}$$

$$
\begin{aligned}
\tilde{D}^{(s)}(a,\beta) &= \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a,\alpha)D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, e^{(s-s')|\alpha|^2/2} D^{(s')}(a,\alpha)D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi}\, e^{-(s'-s)|\alpha|^2/2} D(\alpha,\beta) \int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma)D(\gamma,\alpha) \\
&= \int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma) \underbrace{\int \frac{d^2\alpha}{\pi}\, e^{-(s'-s)|\alpha|^2/2} D(\alpha,\beta-\gamma)}_{=\frac{2}{s'-s}e^{-2|\beta-\gamma|^2/(s'-s)}} \\
&= \frac{2}{s'-s} \int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma) e^{-2|\gamma-\beta|^2/(s'-s)}\ , \quad s \le s'
\end{aligned} \tag{124}
$$

$$
\mathrm{tr}\big(\tilde{D}^{(-s)}(a,\beta)[(a^\dagger)^k a^l]_{(s)}\big) = \int \frac{d^2\gamma}{\pi}\, \underbrace{\mathrm{tr}\big(\tilde{D}^{(-s)\dagger}(a,\beta)\tilde{D}^{(s)}(a,\gamma)\big)}_{=\pi\delta(\beta-\gamma)} (\gamma^*)^k \gamma^l = (\beta^*)^k \beta^l \tag{125}
$$

$$
\mathrm{tr}\big(\tilde{D}^{(-s)}(a,\beta)D^{(s)}(a,\alpha)\big) = \int \frac{d^2\gamma}{\pi}\, \underbrace{\mathrm{tr}\big(\tilde{D}^{(-s)\dagger}(a,\beta)\tilde{D}^{(s)}(a,\gamma)\big)}_{=\pi\delta(\beta-\gamma)} D(\gamma,\alpha) = D(\beta,\alpha) \tag{126}
$$

$$
\begin{aligned}
\tilde{D}^{(s)}(a,\beta) &= \int \frac{d^2\alpha}{\pi}\, D^{(s)}(a-\beta,\alpha) \\
&= D(a,\beta)\left(\int \frac{d^2\alpha}{\pi}\, D^{(s)}(a,\alpha)\right) D^\dagger(a,\beta) \\
&= D(a,\beta)\tilde{D}^{(s)}(a,0)D^\dagger(a,\beta)
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
\tilde{D}^{(s)}(a,\beta+\gamma) &= D(a,\beta+\gamma)\tilde{D}^{(s)}(a,0)D^\dagger(a,\beta+\gamma) \\
&= D(a,\beta)D(a,\gamma)\tilde{D}^{(s)}(a,0)D^\dagger(a,\gamma)D^\dagger(a,\beta) \\
&= D(a,\beta)\tilde{D}^{(s)}(a,\gamma)D^\dagger(a,\beta)
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
\tilde{D}^{(s)}(a,\beta) &= \frac{2}{s'-s} \int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma) e^{-2|\gamma-\beta|^2/(s'-s)} \\
&= \frac{2}{s'-s} \int \frac{d^2\gamma}{\pi}\, \tilde{D}^{(s')}(a,\gamma+\beta) e^{-2|\gamma|^2/(s'-s)} \\
&= \frac{2}{s'-s} \int \frac{d^2\gamma}{\pi}\, D(a,\gamma)\tilde{D}^{(s')}(a,\beta)D^\dagger(a,\gamma) e^{-2|\gamma|^2/(s'-s)}\ , \quad s \le s'
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
\langle\alpha|\tilde{D}^{(s)}(a,0)|\beta\rangle &= \int \frac{d^2\gamma}{\pi}\,\langle\alpha|D^{(s)}(a,\gamma)|\beta\rangle \\
&= \langle\alpha|\beta\rangle \int \frac{d^2\gamma}{\pi}\, e^{-(1-s)|\gamma|^2/2} e^{\gamma\alpha^*-\gamma^*\beta} \\
&= \langle\alpha|\beta\rangle \frac{2}{1-s} e^{-2\beta\alpha^*/(1-s)}\ , \quad s<1
\end{aligned}
\tag{130}
$$

$$
s=0\colon\ \begin{cases}
\langle\alpha|\tilde{D}^{(0)}(a,0)|\beta\rangle = \langle\alpha|\beta\rangle 2e^{-2\beta\alpha^*} = 2\langle\alpha|{-\beta}\rangle \quad\Longrightarrow\quad \tilde{D}^{(0)}(a,0)=2P \\
\tilde{D}^{(0)}(a,\beta) = 2D(a,\beta)PD^\dagger(a,\beta) = 2PD^\dagger(a,2\beta) = 2PD(a,-2\beta) \\
\tilde{D}^{(0)}(a,\beta) = 2\displaystyle\int \frac{d^2\alpha}{\pi} D(a,\beta)|\alpha\rangle\langle-\alpha|D^\dagger(a,\beta) = 2\int \frac{d^2\alpha}{\pi}|\beta+\alpha\rangle\langle\beta-\alpha|D(\alpha,\beta)
\end{cases}
\tag{131}
$$

$$
\begin{aligned}
\langle x|\tilde{D}^{(0)}(a,\beta)|x'\rangle = \langle x|2PD(a,-2\beta)|x'\rangle &= 2\langle -x|D(a,-2\beta)|x'\rangle \\
&= e^{-2i\beta_1\beta_2} e^{2i\beta_2 x}\delta\big(\beta_1-(x+x')/2\big) \\
&= e^{i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big)
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
\langle p|\tilde{D}^{(0)}(a,\beta)|p'\rangle = \langle p|2PD(a,-2\beta)|p'\rangle &= 2\langle -p|D(a,-2\beta)|p'\rangle \\
&= e^{2i\beta_1\beta_2} e^{-2i\beta_1 p}\delta\big(\beta_2-(p+p')/2\big) \\
&= e^{-i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big)
\end{aligned}
\tag{133}
$$

$$
s=-1\colon\ \begin{cases}
\langle\alpha|\tilde{D}^{(-1)}(a,0)|\beta\rangle = \langle\alpha|\beta\rangle e^{-\beta\alpha^*} = e^{-|\alpha|^2/2}e^{-|\beta|^2/2} = \langle\alpha|0\rangle\langle 0|\beta\rangle \\
\quad\Longrightarrow\quad \tilde{D}^{(-1)}(a,0) = |0\rangle\langle 0| \\
\tilde{D}^{(-1)}(a,\beta) = D(a,\beta)|0\rangle\langle 0|D^\dagger(a,\beta) = |\beta\rangle\langle\beta| \\
|\beta\rangle\langle\beta| = \displaystyle\int \frac{d^2\alpha}{\pi}\, e^{-\alpha^* a}e^{\alpha a^\dagger}D(\alpha,\beta) \qquad e^{-\alpha^* a}e^{\alpha a^\dagger} = \int \frac{d^2\beta}{\pi}\,|\beta\rangle\langle\beta|D(\beta,\alpha)
\end{cases}
\tag{134}
$$

12. Operators and associated functions

$$
A = \mathbf{I}|A) = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha) \underbrace{\mathrm{tr}\big(AD^{(s)}(a,\alpha)\big)}_{\equiv F_A^{(s)}(\alpha)} = \int \frac{d^2\alpha}{\pi}\, D^{(-s)\dagger}(a,\alpha)F_A^{(s)}(\alpha)
\tag{135}
$$

$$
F_A^{(s)}(\alpha) = F_{A^\dagger}^{(s)*}(-\alpha) = e^{s|\alpha|^2/2}F_A^{(0)}(\alpha)
\tag{136}
$$

$$
F_A^{(s)}(0) = \mathrm{tr}(A)
\tag{137}
$$

$$
F_P^{(s)}(\alpha) = e^{s|\alpha|^2/2}F_P^{(0)}(\alpha) = \frac{1}{2}e^{s|\alpha|^2/2}
\tag{138}
$$

$$
\begin{aligned}
\operatorname{tr}(A^\dagger B) &= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,\underbrace{\operatorname{tr}\big(D^{(s)}(a,\alpha)D^{(-s)\dagger}(a,\beta)\big)}_{=\pi\delta(\beta-\alpha)} F_A^{(-s)*}(\alpha)F_B^{(s)}(\beta) \\
&= \int \frac{d^2\alpha}{\pi}\,F_A^{(-s)*}(\alpha)F_B^{(s)}(\alpha)
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
A^\dagger B &= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,D^{(s)}(a,\alpha)D^{(-s)\dagger}(a,\beta)F_A^{(-s)*}(\alpha)F_B^{(s)}(\beta) \\
&= \int \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,D(\beta,-\alpha/2)D^\dagger(a,\beta-\alpha)F_A^{(-s)*}(\alpha)F_B^{(s)}(\beta) \\
&= \int \frac{d^2\mu}{\pi}\,D^\dagger(a,\mu)\underbrace{\int \frac{d^2\nu}{\pi}\,D(\nu,\mu/2)F_A^{(-s)*}(\nu-\mu/2)F_B^{(s)}(\nu+\mu/2)}_{=F^{(0)}_{A^\dagger B}(\mu)}
\end{aligned}
\tag{140}
$$

$$
A = \mathbf{I}|A) = \int \frac{d^2\beta}{\pi}\,\tilde D^{(-s)}(a,\beta)\underbrace{\operatorname{tr}\big(A\tilde D^{(s)}(a,\beta)\big)}_{\equiv \tilde F_A^{(s)}(\beta)} = \int \frac{d^2\beta}{\pi}\,\tilde D^{(-s)}(a,\beta)\tilde F_A^{(s)}(\beta)
\tag{141}
$$

$$
\tilde F_A^{(s)}(\beta) = \tilde F_{A^\dagger}^{(s)*}(\beta)
\tag{142}
$$

$$
\operatorname{tr}(A) = \int \frac{d^2\beta}{\pi}\,\tilde F_A^{(s)}(\beta)
\tag{143}
$$

$$
\operatorname{tr}(A^\dagger B) = \int \frac{d^2\beta}{\pi}\,\frac{d^2\gamma}{\pi}\,\underbrace{\operatorname{tr}\big(\tilde D^{(s)}(a,\beta)\tilde D^{(-s)}(a,\gamma)\big)}_{=\pi\delta(\gamma-\beta)}\tilde F_A^{(-s)*}(\beta)\tilde F_B^{(s)}(\gamma) = \int \frac{d^2\beta}{\pi}\,\tilde F_A^{(-s)*}(\beta)\tilde F_B^{(s)}(\beta)
\tag{144}
$$

$$
\begin{aligned}
A^\dagger B &= \int \frac{d^2\beta}{\pi}\,\frac{d^2\gamma}{\pi}\,\tilde F_A^{(-s)*}(\beta)\tilde F_B^{(s)}(\gamma)\,\underbrace{\tilde D^{(s)}(a,\beta)\tilde D^{(-s)}(a,\gamma)}_{=4D^\dagger\left(a,2(\gamma-\beta)\right)D(\beta,2\gamma)} \\
&= \int \frac{d^2\mu}{\pi}\,D^\dagger(a,\mu)\underbrace{\int \frac{d^2\nu}{\pi}\,D(\nu,\mu)\tilde F_A^{(-s)*}(\nu-\mu/4)\tilde F_B^{(s)}(\nu+\mu/4)}_{=F^{(0)}_{A^\dagger B}(\mu)}
\end{aligned}
\tag{145}
$$

$$
\tilde F_A^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi}\,F_A^{(s)}(\alpha)D(\alpha,\beta) \qquad F_A^{(s)}(\alpha) = \int \frac{d^2\beta}{\pi}\,\tilde F_A^{(s)}(\beta)D(\beta,\alpha)
\tag{146}
$$

$$
\begin{aligned}
\tilde F_A^{(s)}(\beta) &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\tilde F_A^{(s')}(\gamma)e^{-2|\gamma-\beta|^2/(s'-s)} \\
&= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\tilde F_A^{(s')}(\beta+\gamma)e^{-2|\gamma|^2/(s'-s)}\,, \quad s\le s'
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
\tilde{F}_A^{(s)}(\beta) &= \mathrm{tr}\big(A\tilde{D}^{(s)}(a,\beta)\big) \\
&= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,\mathrm{tr}\big(AD(a,\gamma)\tilde{D}^{(s')}(a,\beta)D^\dagger(a,\gamma)\big)e^{-2|\gamma|^2/(s'-s)} \\
&= \mathrm{tr}\Bigg(\underbrace{\bigg(\frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,D^\dagger(a,\gamma)AD(a,\gamma)e^{-2|\gamma|^2/(s'-s)}\bigg)}_{=A'}\tilde{D}^{(s')}(a,\beta)\Bigg) \\
&= \tilde{F}_{A'}^{(s')}(\beta)\;, \quad s \le s'
\end{aligned} \tag{148}
$$

$$
A = \sum_{k,l} f_{kl}^{(-s)}[(a^\dagger)^k a^l]_{(-s)} \quad \Longleftrightarrow \quad \tilde{F}_A^{(s)}(\beta) = \sum_{k,l} f_{kl}^{(-s)}(\beta^*)^k\beta^l \tag{149}
$$

$$
s=+1\text{:}\;\begin{cases} A = \displaystyle\int \frac{d^2\beta}{\pi}\,\tilde{D}^{(-1)}(a,\beta)\tilde{F}_A^{(+1)}(\beta) = \int \frac{d^2\beta}{\pi}\,|\beta\rangle\langle\beta|\tilde{F}_A^{(+1)}(\beta) \\ \tilde{F}_A^{(+1)}(\beta) = \displaystyle\int \frac{d^2\alpha}{\pi}\,F_A^{(+1)}(\alpha)D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\big(Ae^{\alpha a^\dagger}e^{-\alpha^* a}\big)D(\alpha,\beta) \end{cases} \tag{150}
$$

$$
s=0\text{:}\;\begin{cases} A = \displaystyle\int \frac{d^2\beta}{\pi}\,\tilde{D}^{(0)}(a,\beta)\tilde{F}_A^{(0)}(\beta) = 2P\int \frac{d^2\beta}{\pi}\,D(a,-2\beta)\tilde{F}_A^{(0)}(\beta) \\ \tilde{F}_A^{(0)}(\beta) = \displaystyle\int \frac{d^2\alpha}{\pi}\,F_A^{(0)}(\alpha)D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\big(AD(a,\alpha)\big)D(\alpha,\beta) \\ \tilde{F}_A^{(0)}(\beta) = \mathrm{tr}\big(A\tilde{D}^{(0)}(a,\beta)\big) = 2\,\mathrm{tr}\big(APD(a,-2\beta)\big) = 2\displaystyle\int \frac{d^2\alpha}{\pi}\langle\beta-\alpha|A|\beta+\alpha\rangle D(\alpha,\beta) \end{cases} \tag{151}
$$

$$
\tilde{F}_P^{(0)}(\beta) = 2\,\mathrm{tr}\big(D(a,-2\beta)\big) = 2\pi\delta(-2\beta) = \frac{\pi}{2}\delta(\beta) \tag{152}
$$

$$
\begin{aligned}
\langle x|A|x'\rangle &= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,\langle x|2PD(a,-2\beta)|x'\rangle \\
&= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,e^{i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big) \\
&= \int \frac{d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}\Big((x+x')/2+i\beta_2\Big)\bigg)\,e^{i\beta_2(x-x')} \\
\tilde{F}_A^{(0)}(\beta) &= 2\int dx\,\langle x|APD(a,-2\beta)|x\rangle \\
&= \int dx\,dx'\,\langle x|A|x'\rangle\langle x'|2PD(a,-2\beta)|x\rangle \\
&= \int dx\,dx'\,\langle x|A|x'\rangle e^{-i\beta_2(x-x')}\delta\big(\beta_1-(x+x')/2\big) \\
&= \int dX\,d\xi\,\langle X+\xi/2|A|X-\xi/2\rangle e^{-i\beta_2\xi}\delta\big(\beta_1-X\big) \\
&= \int d\xi\,\langle \beta_1+\xi/2|A|\beta_1-\xi/2\rangle e^{-i\beta_2\xi}
\end{aligned} \tag{153}
$$

$$\begin{aligned}\langle p|A|p'\rangle &= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,\langle p|2PD(a,-2\beta)|p'\rangle \\ &= \int \frac{d\beta_1\,d\beta_2}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}(\beta_1+i\beta_2)\bigg)\,e^{-i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big) \\ &= \int \frac{d\beta_1}{2\pi}\,\tilde{F}_A^{(0)}\bigg(\frac{1}{\sqrt{2}}\Big(\beta_1+i(p+p')/2\Big)\bigg)\,e^{-i\beta_1(p-p')} \\ \tilde{F}_A^{(0)}(\beta) &= 2\int dp\,\langle p|APD(a,-2\beta)|p\rangle \\ &= \int dp\,dp'\,\langle p|A|p'\rangle\langle p'|2PD(a,-2\beta)|p\rangle \\ &= \int dp\,dp'\,\langle p|A|p'\rangle e^{i\beta_1(p-p')}\delta\big(\beta_2-(p+p')/2\big) \\ &= \int dP\,d\eta\,\langle P+\eta/2|A|P-\eta/2\rangle e^{i\beta_1\eta}\delta\big(\beta_2-P\big) \\ &= \int d\eta\,\langle \beta_2+\eta/2|A|\beta_2-\eta/2\rangle e^{i\beta_1\eta}\end{aligned} \tag{154}$$

$$s=-1\colon\ \begin{cases} A = \displaystyle\int \frac{d^2\beta}{\pi}\,\tilde{D}^{(+1)}(a,\beta)\tilde{F}_A^{(-1)}(\beta) \\ \tilde{F}_A^{(-1)}(\beta) = \mathrm{tr}\big(A\tilde{D}^{(-1)}(a,\beta)\big) = \langle\beta|A|\beta\rangle \end{cases} \tag{155}$$

$$\tilde{F}_P^{(-1)}(\beta) = \langle\beta|P|\beta\rangle = \langle\beta|-\beta\rangle = e^{-2|\beta|^2} \tag{156}$$

13. Characteristic functions and quasiprobability distributions

$$\begin{pmatrix} s\text{-ordered} \\ \text{characteristic} \\ \text{function}\end{pmatrix} = \Phi_\rho^{(s)}(\alpha) \equiv F_\rho^{(s)}(\alpha) = \mathrm{tr}\big(\rho D^{(s)}(a,\alpha)\big) = \Phi_\rho^{(s)*}(-\alpha) = e^{s|\alpha|^2/2}\Phi_\rho^{(0)}(\alpha) \tag{157}$$

$$\rho = \int \frac{d^2\alpha}{\pi}\,D^{(-s)\dagger}(a,\alpha)\Phi_\rho^{(s)}(\alpha) \tag{158}$$

$$\Phi_\rho^{(s)}(\alpha) = \sum_{k,l}\frac{\alpha^k(-\alpha^*)^l}{k!\,l!}\mathrm{tr}\Big(\rho[(a^\dagger)^k a^l]_{(s)}\Big) \tag{159}$$

$$\Phi_\rho^{(s)}(0) = \mathrm{tr}(\rho) = 1 \tag{160}$$

$$\mathrm{tr}(\rho_1\rho_2) = \int \frac{d^2\alpha}{\pi}\,\Phi_{\rho_1}^{(-s)*}(\alpha)\Phi_{\rho_2}^{(s)}(\alpha) \tag{161}$$

$$\mathrm{tr}(\rho^2) = \int \frac{d^2\alpha}{\pi}\,\Phi_{\rho}^{(-s)*}(\alpha)\Phi_{\rho}^{(s)}(\alpha) \tag{162}$$

$$|\Phi_\rho^{(s)}(\alpha)| \le e^{s|\alpha|^2/2} \tag{163}$$

$$\begin{pmatrix} s\text{-ordered} \\ \text{quasiprobability} \\ \text{distribution} \end{pmatrix} = W_\rho^{(s)}(\beta) \equiv \frac{1}{\pi}\tilde{F}_\rho^{(s)}(\beta) = \frac{1}{\pi}\mathrm{tr}\big(\rho\tilde{D}^{(s)}(a,\beta)\big) = W_\rho^{(s)*}(\beta) \tag{164}$$

$$\rho = \int d^2\beta\, \tilde{D}^{(-s)}(a,\beta)W_\rho^{(s)}(\beta) \tag{165}$$

$$\mathrm{tr}\Big(\rho[(a^\dagger)^k a^l]_{(s)}\Big) = \int d^2\beta\,\mathrm{tr}\Big(\tilde{D}^{(-s)}(a,\beta)[(a^\dagger)^k a^l]_{(s)}\Big)W_\rho^{(s)}(\beta) = \int d^2\beta\,(\beta^*)^k\beta^l W_\rho^{(s)}(\beta) \tag{166}$$

$$\mathrm{tr}(\rho A) = \int \frac{d^2\beta}{\pi}\,\tilde{F}_\rho^{(-s)*}(\beta)\tilde{F}_A^{(s)}(\beta) = \int d^2\beta\,\tilde{F}_A^{(s)}(\beta)W_\rho^{(-s)}(\beta) \tag{167}$$

$$1 = \mathrm{tr}(\rho) = \int d^2\beta\, W_\rho^{(s)}(\beta) \tag{168}$$

$$\mathrm{tr}(\rho_1\rho_2) = \pi\int d^2\beta\, W_{\rho_1}^{(-s)}(\beta)W_{\rho_2}^{(s)}(\beta) \tag{169}$$

$$\mathrm{tr}(\rho^2) = \pi\int d^2\beta\, W_\rho^{(-s)}(\beta)W_\rho^{(s)}(\beta) \tag{170}$$

$$W_\rho^{(s)}(\beta) = \int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(s)}(\alpha)D(\alpha,\beta) \qquad \Phi_\rho^{(s)}(\alpha) = \int d^2\beta\, W_\rho^{(s)}(\beta)D(\beta,\alpha) \tag{171}$$

$$\begin{aligned} W_\rho^{(s)}(\beta) &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,W_\rho^{(s')}(\gamma)e^{-2|\gamma-\beta|^2/(s'-s)} \\ &= \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,W_\rho^{(s')}(\beta+\gamma)e^{-2|\gamma|^2/(s'-s)}\ , \quad s\le s' \end{aligned} \tag{172}$$

$$\begin{aligned} &W_\rho^{(s)}(\beta) = W_{\rho'}^{(s')}(\beta)\ ,\ \text{for } s\le s',\ \text{where} \\ &\qquad \rho' = \frac{2}{s'-s}\int \frac{d^2\gamma}{\pi}\,D^\dagger(a,\gamma)\rho D(a,\gamma)e^{-2|\gamma|^2/(s'-s)} \end{aligned} \tag{173}$$

$$\begin{aligned} F_{\rho^2}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi}\,D(\nu,\mu/2)\Phi_\rho^{(-s)*}(\nu-\mu/2)\Phi_\rho^{(s)}(\nu+\mu/2) \\ &= \pi\int d^2\nu\, D(\nu,\mu)W_\rho^{(-s)}(\nu-\mu/4)W_\rho^{(s)}(\nu+\mu/4) \end{aligned} \tag{174}$$

$$\begin{aligned} \Phi_{|\psi\rangle\langle\psi|}^{(0)}(\mu) &= \int \frac{d^2\nu}{\pi}\,D(\nu,\mu/2)\Phi_{|\psi\rangle\langle\psi|}^{(-s)*}(\nu-\mu/2)\Phi_{|\psi\rangle\langle\psi|}^{(s)}(\nu+\mu/2) \\ &= \pi\int d^2\nu\, D(\nu,\mu)W_{|\psi\rangle\langle\psi|}^{(-s)}(\nu-\mu/4)W_{|\psi\rangle\langle\psi|}^{(s)}(\nu+\mu/4) \end{aligned} \tag{175}$$

$$s=+1\!: \begin{cases} \dfrac{1}{\pi}\tilde{F}_\rho^{(+1)}(\beta) = W_\rho^{(+1)}(\beta) \equiv P(\beta) = \begin{pmatrix} \text{Glauber} \\ P\ \text{function} \end{pmatrix} \\ \rho = \displaystyle\int d^2\beta\,\tilde{D}^{(-1)}(a,\beta)P(\beta) = \int d^2\beta\, P(\beta)|\beta\rangle\langle\beta| \\ P(\beta) = \displaystyle\int \frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(+1)}(\alpha)D(\alpha,\beta) = \int \frac{d^2\alpha}{\pi^2}\,\mathrm{tr}\big(\rho e^{\alpha a^\dagger}e^{-\alpha^* a}\big)D(\alpha,\beta) \end{cases} \tag{176}$$

$$
s=0:\quad \begin{cases}
\dfrac{1}{\pi}\tilde{F}_\rho^{(0)}(\beta)=W_\rho^{(0)}(\beta)\equiv W(\beta)=\begin{pmatrix}\text{Wigner}\\ \text{function}\end{pmatrix}\\
\rho=\displaystyle\int d^2\beta\,\tilde{D}^{(0)}(a,\beta)W(\beta)=2P\int d^2\beta\,D(a,-2\beta)W(\beta)\\
W(\beta)=\displaystyle\int\frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(0)}(\alpha)D(\alpha,\beta)=\int\frac{d^2\alpha}{\pi^2}\,\mathrm{tr}\big(\rho D(a,\alpha)\big)D(\alpha,\beta)\\
W(\beta)=\dfrac{1}{\pi}\mathrm{tr}\big(\rho\tilde{D}^{(0)}(a,\beta)\big)=\underbrace{\dfrac{2}{\pi}\mathrm{tr}\big(\rho D(a,\beta)PD^\dagger(a,\beta)\big)}_{\le 2/\pi}=\dfrac{2}{\pi}\mathrm{tr}\big(\rho PD(a,-2\beta)\big)\\
\qquad\;=\dfrac{2}{\pi^2}\displaystyle\int d^2\alpha\,\langle\beta-\alpha|\rho|\beta+\alpha\rangle D(\alpha,\beta)\\
\mathrm{tr}(\rho_1\rho_2)=\pi\displaystyle\int d^2\beta\,W_{\rho_1}(\beta)W_{\rho_2}(\beta)\qquad \mathrm{tr}(\rho^2)=\pi\int d^2\beta\,W^2(\beta)
\end{cases}
\tag{177}
$$

If position β_1 and momentum β_2 are used as the variables in the Wigner function, it is conventional to use a rescaled Wigner function defined by

$$
W'(\beta_1,\beta_2)=\frac{1}{2}W(\beta)=\int\frac{d\alpha_1\,d\alpha_2}{(2\pi)^2}\,\mathrm{tr}\big(\rho e^{i(\alpha_2x-\alpha_1p)}\big)e^{i(\beta_2\alpha_1-\beta_1\alpha_2)}\,. \tag{178}
$$

$$
\begin{aligned}
\langle x|\rho|x'\rangle&=\int\frac{d\beta_2}{2}\,W\bigg(\frac{1}{\sqrt{2}}\Big((x+x')/2+i\beta_2\Big)\bigg)\,e^{i\beta_2(x-x')}\\
&=\int d\beta_2\,W'\big((x+x')/2,\beta_2\big)e^{i\beta_2(x-x')}
\end{aligned}
\tag{179}
$$

$$
W'(\beta_1,\beta_2)=\frac{1}{2}W(\beta)=\int\frac{d\xi}{2\pi}\,\langle\beta_1+\xi/2|\rho|\beta_1-\xi/2\rangle e^{-i\beta_2\xi}
$$

$$
\begin{aligned}
\langle p|\rho|p'\rangle&=\int\frac{d\beta_1}{2}\,W\bigg(\frac{1}{\sqrt{2}}\Big(\beta_1+i(p+p')/2\Big)\bigg)\,e^{-i\beta_1(p-p')}\\
&=\int d\beta_1\,W'\big(\beta_1,(p+p')/2\big)e^{-i\beta_1(p-p')}
\end{aligned}
\tag{180}
$$

$$
W'(\beta_1,\beta_2)=\frac{1}{2}W(\beta)=\int\frac{d\eta}{2\pi}\,\langle\beta_2+\eta/2|\rho|\beta_2-\eta/2\rangle e^{i\beta_1\eta}
$$

$$
1=\mathrm{tr}(\rho)=\int d^2\beta\,W(\beta)=\int d\beta_1\,d\beta_2\,W'(\beta_1,\beta_2) \tag{181}
$$

$$
\mathrm{tr}(\rho_1\rho_2)=2\pi\int d\beta_1\,d\beta_2\,W'_{\rho_1}(\beta_1,\beta_2)W'_{\rho_2}(\beta_1,\beta_2)\qquad \mathrm{tr}(\rho^2)=2\pi\int d\beta_1\,d\beta_2\,W'^2(\beta_1,\beta_2) \tag{182}
$$

$$
s=-1:\quad\begin{cases}
\dfrac{1}{\pi}\tilde{F}_\rho^{(-1)}(\beta)=W_\rho^{(-1)}(\beta)\equiv Q(\beta)=\begin{pmatrix}\text{Husimi}\\ Q\text{ function}\end{pmatrix}\\
\rho=\displaystyle\int d^2\beta\,\tilde{D}^{(+1)}(a,\beta)Q(\beta)\\
Q(\beta)=\dfrac{1}{\pi}\mathrm{tr}\big(\rho\tilde{D}^{(-1)}(a,\beta)\big)=\dfrac{1}{\pi}\langle\beta|\rho|\beta\rangle\le\dfrac{1}{\pi}
\end{cases}
\tag{183}
$$

$$\Phi^{(s)}_{|\gamma\rangle\langle\gamma|}(\alpha) = \langle\gamma|D^{(s)}(a,\alpha)|\gamma\rangle = e^{(s-1)|\alpha|^2/2}D(\gamma,\alpha) \tag{184}$$

$$\begin{aligned} W^{(s)}_{|\gamma\rangle\langle\gamma|}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{|\gamma\rangle\langle\gamma|}(\alpha)D(\alpha,\beta) \\ &= \int \frac{d^2\alpha}{\pi^2}\,e^{(s-1)|\alpha|^2/2}D(\alpha,\beta-\gamma) \\ &= \frac{2}{\pi(1-s)}e^{-2|\beta-\gamma|^2/(1-s)} \end{aligned} \tag{185}$$

$$\rho = |\gamma\rangle\langle\gamma|\text{:}\ \begin{cases} s=+1\text{:}\quad P(\beta) = \delta(\beta-\gamma) \\ s=0\text{:}\quad W(\beta) = \dfrac{2}{\pi}\,e^{-2|\beta-\gamma|^2}\ ,\quad W'(\beta_1,\beta_2) = \dfrac{1}{\pi}\,e^{-(\beta_1-\gamma_1)^2-(\beta_2-\gamma_2)^2} \\ s=-1\text{:}\quad Q(\beta) = \dfrac{1}{\pi}\,e^{-|\beta-\gamma|^2} = \dfrac{1}{\pi}\,|\langle\beta|\gamma\rangle|^2 \end{cases} \tag{186}$$

$$\begin{aligned} \rho = |\gamma\rangle\langle\gamma|:\quad W(\beta) &= \frac{2}{\pi}\langle\gamma|D^\dagger(a,-\beta)PD(a,-\beta)|\gamma\rangle \\ &= \frac{2}{\pi}\langle\gamma-\beta|P|\gamma-\beta\rangle \\ &= \frac{2}{\pi}\langle\gamma-\beta|\beta-\gamma\rangle \\ &= \frac{2}{\pi}\,e^{-2|\beta-\gamma|^2} \end{aligned} \tag{187}$$

14. Thermal states

$$\rho = \frac{1}{Z}e^{-\lambda a^\dagger a} = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\lambda n}|n\rangle\langle n| \tag{188}$$

$$Z = \mathrm{tr}\big(e^{-\lambda a^\dagger a}\big) = \sum_{n=0}^{\infty} e^{-\lambda n} = \frac{1}{1-e^{-\lambda}} \tag{189}$$

$$\bar{n} = \mathrm{tr}(\rho a^\dagger a) = \frac{1}{Z}\sum_{n=0}^{\infty} ne^{-\lambda n} = -\frac{1}{Z}\frac{\partial Z}{\partial\lambda} = \frac{1}{e^\lambda - 1} \tag{190}$$

$$\begin{aligned} &e^\lambda = \frac{1+\bar{n}}{\bar{n}} \\ &Z = 1+\bar{n} = \frac{1}{1-e^{-\lambda}} \\ &\rho = \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^{a^\dagger a} = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n|n\rangle\langle n| \end{aligned} \tag{191}$$

$$\begin{aligned} &\overline{n^2} = \mathrm{tr}\big(\rho(a^\dagger a)^2\big) = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 e^{-\lambda n} = \frac{1}{Z}\frac{\partial^2 Z}{\partial\lambda^2} = \frac{1}{e^\lambda-1} + \frac{2}{(e^\lambda-1)^2} = \bar{n}+2\bar{n}^2 \\ &(\Delta n)^2 = \overline{n^2} - \bar{n}^2 = \bar{n}^2 + \bar{n} \end{aligned} \tag{192}$$

$$S=-\mathrm{tr}(\rho\ln\rho)=\ln Z+\lambda\bar n=(1+\bar n)\ln(1+\bar n)-\bar n\ln\bar n \tag{193}$$

$$\begin{aligned}\Big\langle n\Big|\Big(\int\frac{d^2\beta}{\pi\bar n}\,e^{-|\beta|^2/\bar n}|\beta\rangle\langle\beta|\Big)\Big|m\Big\rangle&=\int\frac{d^2\beta}{\pi\bar n}\,e^{-|\beta|^2/\bar n}\langle n|\beta\rangle\langle\beta|m\rangle\\&=\frac{1}{\sqrt{n!\,m!}}\int\frac{d^2\beta}{\pi\bar n}\,e^{-|\beta|^2(1+\bar n)/\bar n}\beta^n(\beta^*)^m\\&=\frac{1}{2\sqrt{n!\,m!}}\int\frac{d|\beta|^2\,d\phi}{\pi\bar n}\,e^{-|\beta|^2(1+\bar n)/\bar n}|\beta|^{n+m}e^{i(n-m)\phi}\\&=\frac{1}{n!}\delta_{nm}\int_0^\infty\frac{d|\beta|^2}{\bar n}\,e^{-|\beta|^2(1+\bar n)/\bar n}|\beta|^{2n}\\&=\frac{1}{n!}\delta_{nm}\frac{\bar n^n}{(1+\bar n)^{n+1}}\int_0^\infty du\,e^{-u}u^n\\&=\delta_{nm}\frac{\bar n^n}{(1+\bar n)^{n+1}}\end{aligned} \tag{194}$$

$$\rho=\int\frac{d^2\beta}{\pi\bar n}\,e^{-|\beta|^2/\bar n}|\beta\rangle\langle\beta|\quad\Longrightarrow\quad P(\beta)=W_\rho^{(+1)}(\beta)=\frac{1}{\pi}\tilde F_\rho^{(+1)}(\beta)=\frac{1}{\pi\bar n}e^{-|\beta|^2/\bar n} \tag{195}$$

$$\Phi_\rho^{(+1)}(\alpha)=F_\rho^{(+1)}(\alpha)=\int d^2\beta\,P(\beta)D(\beta,\alpha)=\int\frac{d^2\beta}{\pi\bar n}\,e^{-|\beta|^2/\bar n}D(\beta,\alpha)=e^{-\bar n|\alpha|^2} \tag{196}$$

$$\Phi_\rho^{(s)}(\alpha)=e^{(s-1)|\alpha|^2/2}\Phi_\rho^{(+1)}(\alpha)=e^{-[\bar n+(1-s)/2]|\alpha|^2} \tag{197}$$

$$\begin{aligned}\frac{1}{\pi}\tilde F_\rho^{(s)}(\beta)=W_\rho^{(s)}(\beta)&=\int\frac{d^2\alpha}{\pi^2}\,\Phi_\rho^{(s)}(\alpha)D(\alpha,\beta)\\&=\int\frac{d^2\alpha}{\pi^2}\,e^{-[\bar n+(1-s)/2]|\alpha|^2}D(\alpha,\beta)\\&=\frac{1}{\pi[\bar n+(1-s)/2]}e^{-|\beta|^2/[\bar n+(1-s)/2]}\end{aligned} \tag{198}$$

$$\begin{aligned}e^{-\bar n|\alpha|^2}&=\Phi_\rho^{(+1)}(\alpha)\\&=\mathrm{tr}\big(\rho D^{(+1)}(a,\alpha)\big)\\&=\frac{1}{1+\bar n}\sum_{n=0}^\infty\left(\frac{\bar n}{1+\bar n}\right)^n\langle n|D^{(+1)}(a,\alpha)|n\rangle\\&=\frac{1}{1+\bar n}\sum_{n=0}^\infty\left(\frac{\bar n}{1+\bar n}\right)^n L_n\big(|\alpha|^2\big)\end{aligned} \tag{199}$$

$$e^{-\lambda a^\dagger a}=\,:\!e^{-(1-e^{-\lambda})a^\dagger a}\!:\,=\,:\!e^{-a^\dagger a/Z}\!:\,=\,:\!e^{-a^\dagger a/(1+\bar n)}\!: \tag{200}$$

$$Q(\beta)=W_\rho^{(-1)}(\beta)=\frac{1}{\pi}\langle\beta|\rho|\beta\rangle=\frac{1}{\pi(1+\bar n)}\langle\beta|\!:\!e^{-a^\dagger a/(1+\bar n)}\!:\!|\beta\rangle=\frac{1}{\pi(1+\bar n)}e^{-|\beta|^2/(1+\bar n)} \tag{201}$$

15. Single-mode squeeze operator and single-mode squeezed states

For additional information, see C. M. Caves and B. L. Schumaker, *Physical Review A* **31**, 3068–3092 (1985) [CS]; B. L. Schumaker and C. M. Caves, *Physical Review A* **31**, 3093–3111 (1985) [SC]; and B. L. Schumaker, *Physics Reports* **135**(6), 317–408 (1986) [S].

$$S(r,\phi) \equiv \exp\Big(\tfrac{1}{2}r\big(a^2e^{-2i\phi} - (a^\dagger)^2e^{2i\phi}\big)\Big) \tag{202}$$

$$S(r,\phi+\pi) = S(r,\phi) \tag{203}$$

$$S^{-1}(r,\phi) = S^\dagger(r,\phi) = S(-r,\phi) = S(r,\phi+\pi/2) \tag{204}$$

$$e^{i\theta a^\dagger a}S(r,\phi)e^{-i\theta a^\dagger a} = S(r,\phi+\theta) \tag{205}$$

$$\begin{aligned} S(r,\phi) &= e^{-\Gamma A^\dagger}e^{-gB}e^{\Gamma^* A} = e^{-\Gamma A^\dagger}e^{\Gamma^* e^{2g}A}e^{-gB} = e^{-gB}e^{-\Gamma e^{2g}A^\dagger}e^{\Gamma^* A} \\ &= e^{\Gamma^* A}e^{-\Gamma e^{2g}A^\dagger}e^{gB} = e^{\Gamma^* A}e^{gB}e^{-\Gamma A^\dagger} = e^{gB}e^{\Gamma^* e^{2g}A}e^{-\Gamma A^\dagger} \end{aligned} \tag{206}$$

$$A \equiv \frac{1}{2}a^2 \qquad B \equiv a^\dagger a + \frac{1}{2} \qquad \Gamma \equiv e^{2i\phi}\tanh r \qquad g \equiv \ln(\cosh r)$$

(See Appendix B of [SC].)

$$\begin{aligned} S^\dagger(r,\phi)aS(r,\phi) &= a\cosh r - a^\dagger e^{2i\phi}\sinh r \\ S^\dagger(r,\phi)a^\dagger S(r,\phi) &= a^\dagger\cosh r - ae^{-2i\phi}\sinh r \end{aligned} \tag{207}$$

$$S^\dagger(r,\phi)D(a,\alpha)S(r,\phi) = D(a\cosh r - a^\dagger e^{2i\phi}\sinh r,\alpha) = D(a,\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r) \tag{208}$$

$$\begin{aligned} S^\dagger(r,\phi)(x\cos\phi + p\sin\phi)S(r,\phi) &= (x\cos\phi + p\sin\phi)e^{-r} \\ S^\dagger(r,\phi)(-x\sin\phi + p\cos\phi)S(r,\phi) &= (-x\sin\phi + p\cos\phi)e^{r} \end{aligned} \tag{209}$$

$$S^\dagger(r,0)xS(r,0) = xe^{-r} \qquad S^\dagger(r,0)pS(r,0) = p\,e^r \tag{210}$$

$$\begin{aligned} &S^\dagger(r',\phi')S(r,\phi) = e^{-i\Theta B}S(R,\Phi) = S(R,\Phi-\Theta)e^{-i\Theta B} \\ &B \equiv a^\dagger a + \frac{1}{2} \\ &e^{i\Theta}\cosh R \equiv \cosh r\cosh r' - e^{2i(\phi-\phi')}\sinh r\sinh r' \\ &e^{i(2\Phi-\Theta)}\sinh R \equiv e^{2i\phi}\sinh r\cosh r' - e^{2i\phi'}\cosh r\sinh r' \end{aligned} \tag{211}$$

(See Appendix B of [SC].)

$$S(r',\phi)S(r,\phi) = S(r+r',\phi) \tag{212}$$

$$\begin{aligned} |\alpha\rangle_{(r,\phi)} &\equiv D(a,\alpha)S(r,\phi)|0\rangle \\ &= D(a,\alpha)|0\rangle_{(r,\phi)} \\ &= S(r,\phi)D(a,\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)|0\rangle \\ &= S(r,\phi)|\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r\rangle \end{aligned} \tag{213}$$

$$e^{-i\theta a^\dagger a}|\alpha\rangle_{(r,\phi)} = |\alpha e^{-i\theta}\rangle_{(r,\phi-\theta)} \tag{214}$$

$$\begin{aligned}(a\cosh r + a^\dagger e^{2i\phi}\sinh r)|\alpha\rangle_{(r,\phi)} &= S(r,\phi)a|\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r\rangle \\ &= (\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)|\alpha\rangle_{(r,\phi)}\end{aligned} \tag{215}$$

$$S^\dagger(r,\phi)D^\dagger(a,\alpha)aD(a,\alpha)S(r,\phi) = a\cosh r - a^\dagger e^{2i\phi}\sinh r + \alpha \tag{216}$$

$$\begin{aligned}S^\dagger(r,\phi)D^\dagger(a,\gamma)D(a,\alpha)D(a,\gamma)S(r,\phi) &= D(a\cosh r - a^\dagger e^{2i\phi}\sinh r + \gamma, \alpha) \\ &= D(\gamma,\alpha)D(a, \alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)\end{aligned} \tag{217}$$

$$\begin{aligned}&{}_{(r,\phi)}\langle\alpha|a|\alpha\rangle_{(r,\phi)} = \alpha \\ &{}_{(r,\phi)}\langle\alpha|a^2|\alpha\rangle_{(r,\phi)} = \alpha^2 - e^{2i\phi}\cosh r\sinh r = \alpha^2 - \frac{1}{2}e^{2i\phi}\sinh 2r \\ &{}_{(r,\phi)}\langle\alpha|a^\dagger a|\alpha\rangle_{(r,\phi)} = |\alpha|^2 + \sinh^2 r = |\alpha|^2 + \frac{1}{2}(\cosh 2r - 1)\end{aligned} \tag{218}$$

$$\begin{aligned}&{}_{(r,\phi)}\langle\alpha|x|\alpha\rangle_{(r,\phi)} = \alpha_1 \\ &{}_{(r,\phi)}\langle\alpha|p|\alpha\rangle_{(r,\phi)} = \alpha_2 \\ &{}_{(r,\phi)}\langle\alpha|(\Delta x)^2|\alpha\rangle_{(r,\phi)} = \frac{1}{2}(\cosh 2r - \sinh 2r\cos 2\phi) = \frac{1}{2}(e^{-2r}\cos^2\phi + e^{2r}\sin^2\phi) \\ &{}_{(r,\phi)}\langle\alpha|(\Delta p)^2|\alpha\rangle_{(r,\phi)} = \frac{1}{2}(\cosh 2r + \sinh 2r\cos 2\phi) = \frac{1}{2}(e^{-2r}\sin^2\phi + e^{2r}\cos^2\phi) \\ &{}_{(r,\phi)}\langle\alpha|\tfrac{1}{2}(\Delta x\Delta p + \Delta p\Delta x)|\alpha\rangle_{(r,\phi)} = -\frac{1}{2}\sinh 2r\sin 2\phi\end{aligned} \tag{219}$$

$$\begin{aligned}|0\rangle_{(r,\phi)} = S(r,\phi)|0\rangle &= \frac{1}{\sqrt{\cosh r}}\exp\left(-\frac{1}{2}(a^\dagger)^2 e^{2i\phi}\tanh r\right)|0\rangle \\ &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!}\sqrt{(2n)!}|2n\rangle \\ &= \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{2}}e^{2i\phi}\tanh r\right)^n\sqrt{\frac{(2n-1)!!}{n!}}|2n\rangle \\ &\qquad [(2n)! = 2^n n!\,(2n-1)!!]\end{aligned} \tag{220}$$

$${}_{(r,\phi)}\langle 0|0\rangle_{(r,\phi)} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\left(\frac{\tanh^2 r}{2}\right)^n\frac{(2n-1)!!}{n!} = \frac{1}{\cosh r\sqrt{1-\tanh^2 r}} = 1 \tag{221}$$

$$\left(\text{Use}\quad (1+x)^{-1/2} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!}\left(\frac{-x}{2}\right)^n\quad .\right)$$

$$\langle x=0|0\rangle_{(r,\phi)} = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!} \underbrace{\sqrt{(2n)!}\langle x=0|2n\rangle}_{=\frac{1}{\pi^{1/4}2^n}H_{2n}(0)}$$

$$= \frac{1}{\pi^{1/4}\sqrt{\cosh r}} \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{2i\phi}\tanh r\right)^n \frac{(2n-1)!!}{n!} \tag{222}$$

$$= \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}}$$

[Use $H_{2n}(0) = (-1)^n (2n)!/n! = (-2)^n (2n-1)!!$: A&S (22.3.10).]

$$0 = \langle x|(a\cosh r + a^\dagger e^{2i\phi}\sinh r)|0\rangle_{(r,\phi)}$$

$$= \frac{1}{\sqrt{2}}\Big((\cosh r + e^{2i\phi}\sinh r)x + (\cosh r - e^{2i\phi}\sinh r)\frac{d}{dx}\Big)\langle x|0\rangle_{(r,\phi)}$$

$$\Longrightarrow \quad \langle x|0\rangle_{(r,\phi)} = \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}} \exp\left(-\frac{1}{2}\frac{\cosh r + e^{2i\phi}\sinh r}{\cosh r - e^{2i\phi}\sinh r}x^2\right) \tag{223}$$

$$= \frac{1}{\pi^{1/4}} \frac{1}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}} \exp\left(-\frac{1}{2}\frac{1 + i\sinh 2r\sin 2\phi}{\cosh 2r - \sinh 2r\cos 2\phi}x^2\right)$$

$$\langle x|\alpha\rangle_{(r,\phi)} = \langle x|D(a,\alpha)|0\rangle_{(r,\phi)}$$

$$= e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}\langle x-\alpha_1|0\rangle_{(r,\phi)} \tag{224}$$

$$= \frac{1}{\pi^{1/4}} \frac{e^{-i\alpha_1\alpha_2/2}e^{i\alpha_2 x}}{(\cosh r - e^{2i\phi}\sinh r)^{1/2}} \exp\left(-\frac{1}{2}\frac{1 + i\sinh 2r\sin 2\phi}{\cosh 2r - \sinh 2r\cos 2\phi}(x-\alpha_1)^2\right)$$

$$\langle \beta|0\rangle_{(r,\phi)} = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}e^{2i\phi}\tanh r)^n}{n!} \underbrace{\sqrt{(2n)!}\langle \beta|2n\rangle}_{=e^{-|\beta|^2/2}(\beta^*)^{2n}}$$

$$= \frac{1}{\sqrt{\cosh r}} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}(\beta^* e^{i\phi})^2\tanh r\right)^n}{n!} \tag{225}$$

$$= \frac{1}{\sqrt{\cosh r}} e^{-|\beta|^2/2} \exp\left(-\frac{1}{2}(\beta^* e^{i\phi})^2\tanh r\right)$$

$$\langle \beta|\alpha\rangle_{(r,\phi)} = \langle \beta|D(a,\alpha)|0\rangle_{(r,\phi)}$$

$$= D(\beta,\alpha/2)\langle \beta-\alpha|0\rangle_{(r,\phi)} \tag{226}$$

$$= \frac{1}{\sqrt{\cosh r}} D(\beta,\alpha/2) e^{-|\beta-\alpha|^2/2} \exp\left(-\frac{1}{2}[(\beta^*-\alpha^*)e^{i\phi}]^2\tanh r\right)$$

$$
\begin{aligned}
|\langle\beta|\alpha\rangle_{(r,\phi)}|^2 &= |\langle\beta-\alpha|0\rangle_{(r,\phi)}|^2 \\
&= \frac{1}{\cosh r}\exp\Bigg[-\bigg(|\beta-\alpha|^2 + \frac{1}{2}\Big([(\beta-\alpha)e^{-i\phi}]^2 + [(\beta^*-\alpha^*)e^{i\phi}]^2\Big)\tanh r\bigg)\Bigg] \\
&= \frac{1}{\cosh r}\exp\Bigg[-\frac{1}{2}\Big((\beta_1-\alpha_1)^2(1+\tanh r\cos 2\phi) \\
&\qquad + (\beta_2-\alpha_2)^2(1-\tanh r\cos 2\phi) \\
&\qquad + 2(\beta_1-\alpha_1)(\beta_2-\alpha_2)\tanh r\sin 2\phi\Big)\Bigg]
\end{aligned}
\tag{227}
$$

$$
\rho_{\alpha;r,\phi} \equiv |\alpha\rangle_{(r,\phi)}\,{}_{(r,\phi)}\langle\alpha| \tag{228}
$$

$$
\begin{aligned}
\Phi^{(s)}_{\rho_{\gamma;r,\phi}}(\alpha) &= {}_{(r,\phi)}\langle\gamma|D^{(s)}(a,\alpha)|\gamma\rangle_{(r,\phi)} \\
&= e^{s|\alpha|^2/2}\langle 0|S^\dagger(r,\phi)D^\dagger(a,\gamma)D(a,\alpha)D(a,\gamma)S(r,\phi)|0\rangle \\
&= D(\gamma,\alpha)e^{s|\alpha|^2/2}\langle 0|D(a,\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r)|0\rangle \\
&= D(\gamma,\alpha)e^{s|\alpha|^2/2}\exp\bigg(-\frac{1}{2}|\alpha\cosh r + \alpha^* e^{2i\phi}\sinh r|^2\bigg) \\
&= D(\gamma,\alpha)\exp\Bigg[-\frac{1}{2}\bigg(|\alpha|^2(-s+\cosh 2r) + \frac{1}{2}[(\alpha e^{-i\phi})^2 + (\alpha^* e^{i\phi})^2]\sinh 2r\bigg)\Bigg] \\
&= D(\gamma,\alpha)\exp\Bigg[-\frac{1}{4}\bigg(\alpha_1^2(-s+\cosh 2r+\sinh 2r\cos 2\phi) \\
&\qquad + \alpha_2^2(-s+\cosh 2r-\sinh 2r\cos 2\phi) \\
&\qquad + 2\alpha_1\alpha_2\sinh 2r\sin 2\phi\bigg)\Bigg] \\
&= D(\gamma,\alpha)\exp\Bigg[-\frac{1}{4}\bigg(\alpha_1^2(-s+e^{-2r}\sin^2\phi + e^{2r}\cos^2\phi) \\
&\qquad + \alpha_2^2(-s+e^{-2r}\cos^2\phi + e^{2r}\sin^2\phi) \\
&\qquad + 2\alpha_1\alpha_2\sinh 2r\sin 2\phi\bigg)\Bigg]
\end{aligned}
\tag{229}
$$

$$
\begin{aligned}
W^{(s)}_{\rho_{0;r,0}}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{\rho_{0;r,0}}(\alpha) D(\alpha,\beta) \\
&= \frac{1}{\pi}\int \frac{d\alpha_1\,d\alpha_2}{2\pi}\, e^{i(\beta_2\alpha_1-\beta_1\alpha_2)} e^{-[\alpha_1^2(-s+e^{2r})+\alpha_2^2(-s+e^{-2r})]/4} \\
&= \frac{2}{\pi}\frac{1}{\sqrt{(-s+e^{-2r})(-s+e^{2r})}}\exp\!\left(-\frac{\beta_1^2}{-s+e^{-2r}}-\frac{\beta_2^2}{-s+e^{2r}}\right) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}}\exp\!\left(-\frac{2|\beta|^2(\cosh 2r-s)+[\beta^2+(\beta^*)^2]\sinh 2r}{1-2s\cosh 2r+s^2}\right)\,, \\
&\qquad\qquad s\le e^{-2|r|}
\end{aligned}
\tag{230}
$$

$$
\begin{aligned}
W^{(s)}_{\rho_{0;r,\phi}}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha) D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi^2}\,\underbrace{\Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha e^{i\phi})}_{=\Phi^{(s)}_{\rho_{0;r,0}}(\alpha)} D(\alpha,\beta e^{-i\phi}) \\
&= W^{(s)}_{\rho_{0;r,0}}(\beta e^{-i\phi}) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad\times\exp\!\left(-\frac{2|\beta|^2(\cosh 2r-s)+[(\beta e^{-i\phi})^2+(\beta^* e^{i\phi})^2]\sinh 2r}{1-2s\cosh 2r+s^2}\right) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad\times\exp\left(\frac{\begin{array}{c}-\beta_1^2(-s+\cosh 2r+\sinh 2r\cos 2\phi)\\ -\beta_2^2(-s+\cosh 2r-\sinh 2r\cos 2\phi)\\ -2\beta_1\beta_2\sinh 2r\sin 2\phi\end{array}}{1-2s\cosh 2r+s^2}\right)\,,\quad s\le e^{-2|r|}
\end{aligned}
\tag{231}
$$

$$
\begin{aligned}
W^{(s)}_{\rho_{\gamma;r,\phi}}(\beta) &= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{\rho_{\gamma;r,\phi}}(\alpha)D(\alpha,\beta) \\
&= \int \frac{d^2\alpha}{\pi^2}\,\Phi^{(s)}_{\rho_{0;r,\phi}}(\alpha)D(\alpha,\beta-\gamma) \\
&= W^{(s)}_{\rho_{0;r,\phi}}(\beta-\gamma) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(-\frac{2|\beta-\gamma|^2(\cosh 2r - s) + \left([(\beta-\gamma)e^{-i\phi}]^2 + [(\beta^*-\gamma^*)e^{i\phi}]^2\right)\sinh 2r}{1-2s\cosh 2r+s^2}\right) \\
&= \frac{2}{\pi}\frac{1}{\sqrt{1-2s\cosh 2r+s^2}} \\
&\quad \times \exp\left(\frac{\begin{array}{c}-(\beta_1-\gamma_1)^2(-s+\cosh 2r+\sinh 2r\cos 2\phi)\\ -(\beta_2-\gamma_2)^2(-s+\cosh 2r-\sinh 2r\cos 2\phi)\\ -2(\beta_1-\gamma_1)(\beta_2-\gamma_2)\sinh 2r\sin 2\phi\end{array}}{1-2s\cosh 2r+s^2}\right), \quad s \le e^{-2|r|}
\end{aligned}
\tag{232}
$$

$$
\rho=\rho_{\gamma;r,\phi}: \begin{cases}
s=0: & \begin{aligned} W(\beta) &= \frac{2}{\pi}\,e^{-2|\beta-\gamma|^2\cosh 2r - ([(\beta-\gamma)e^{-i\phi}]^2+[(\beta^*-\gamma^*)e^{i\phi}]^2)\sinh 2r} \\ W'(\beta_1,\beta_2) &= \frac{1}{\pi}\,\exp\begin{pmatrix} -(\beta_1-\gamma_1)^2(\cosh 2r+\sinh 2r\cos 2\phi) \\ -(\beta_2-\gamma_2)^2(\cosh 2r-\sinh 2r\cos 2\phi) \\ -2(\beta_1-\gamma_1)(\beta_2-\gamma_2)\sinh 2r\sin 2\phi \end{pmatrix} \end{aligned} \\
s=-1: & \begin{aligned} Q(\beta) &= \frac{1}{\pi\cosh r}e^{-|\beta-\gamma|^2-\frac{1}{2}([(\beta-\gamma)e^{-i\phi}]^2+[(\beta^*-\gamma^*)e^{i\phi}]^2)\tanh r} \\ &= \frac{1}{\pi\cosh r}\,\exp\begin{pmatrix} -\frac{1}{2}(\beta_1-\gamma_1)^2(1+\tanh r\cos 2\phi) \\ -\frac{1}{2}(\beta_2-\gamma_2)^2(1-\tanh r\cos 2\phi) \\ -(\beta_1-\gamma_1)(\beta_2-\gamma_2)\tanh r\sin 2\phi \end{pmatrix} \\ &= \frac{1}{\pi}\,|\langle\beta|\gamma\rangle_{(r,\phi)}|^2 \end{aligned}
\end{cases}
\tag{233}
$$

$$
\begin{aligned}
\rho=\rho_{0;r,0}\ :\quad W(\beta) &= \frac{2}{\pi}\,{}_{(r,0)}\langle 0|PD(a,-2\beta)|0\rangle_{(r,\phi)} \\
&= \frac{2}{\pi}\,{}_{(r,0)}\langle 0|D(a,-2\beta)|0\rangle_{(r,\phi)} \\
&= \frac{2}{\pi}\Phi^{(0)}_{\rho_{0;r,0}}(-2\beta) \\
&= \frac{2}{\pi}e^{-2|\beta|^2\cosh 2r-[\beta^2+(\beta^*)^2]\sinh 2r}
\end{aligned}
\tag{234}
$$