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I. General considerations

An *antilinear operator* $\mathcal{K} : |\psi\rangle \to \mathcal{K}|\psi\rangle$ acts on linear combinations according to

$$\mathcal{K}\big(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle\big) = \alpha_1^*\big(\mathcal{K}|\psi_1\rangle\big) + \alpha_2^*\big(\mathcal{K}|\psi_2\rangle\big) . \tag{1}$$

A product of linear and antilinear operators is linear if it has an even number of antilinears and antilinear if it has an odd number of antilinears.

The left action of an antilinear operator, i.e., $\langle\phi| \to \langle\phi|\mathcal{K}$, is given by

$$\big(\langle\phi|\mathcal{K}\big)|\psi\rangle = \big[\langle\phi|\big(\mathcal{K}|\psi\rangle\big)\big]^* . \tag{2}$$

The complex conjugation is present so that

$$\begin{aligned}\big[\big(\langle\phi_1|\alpha_1^* + \langle\phi_2|\alpha_2^*\big)\mathcal{K}\big]|\psi\rangle &= \big[\big(\langle\phi_1|\alpha_1^* + \langle\phi_2|\alpha_2^*\big)\big(\mathcal{K}|\psi\rangle\big)\big]^* \\ &= \alpha_1\big[\langle\phi_1|\big(\mathcal{K}|\psi\rangle\big)\big]^* + \alpha_2\big[\langle\phi_2|\big(\mathcal{K}|\psi\rangle\big)\big]^* \\ &= \alpha_1\big(\langle\phi_1|\mathcal{K}\big)|\psi\rangle + \alpha_2\big(\langle\phi_2|\mathcal{K}\big)|\psi\rangle \\ &= \big[\big(\langle\phi_1|\mathcal{K}\big)\alpha_1\big]|\psi\rangle + \big[\big(\langle\phi_2|\mathcal{K}\big)\alpha_2\big]|\psi\rangle \\ &= \big[\big(\langle\phi_1|\mathcal{K}\big)\alpha_1 + \big(\langle\phi_2|\mathcal{K}\big)\alpha_2]|\psi\rangle ,\end{aligned} \tag{3}$$

i.e., so that $\mathcal{K}$ is antilinear to the left:

$$\big(\langle\phi_1|\alpha_1^* + \langle\phi_2|\alpha_2^*\big)\mathcal{K} = \big(\langle\phi_1|\mathcal{K}\big)\alpha_1 + \big(\langle\phi_2|\mathcal{K}\big)\alpha_2 . \tag{4}$$

The presence of the complex conjugation in Eq. (2) means that in a matrix element one must always indicate explicitly whether an antilinear operator acts to the right or to the left.

The adjoint (Hermitian conjugate) of an antilinear operator is defined in the same way as for a linear operator, i.e.,

$$\big(\mathcal{K}|\psi\rangle\big)^\dagger = \big(\langle\psi|\mathcal{K}^\dagger\big) , \tag{5}$$

but this means that

$$\langle\phi|\big(\mathcal{K}|\psi\rangle\big) = \big[\big(\langle\psi|\mathcal{K}^\dagger\big)|\phi\rangle\big]^* = \langle\psi|\big(\mathcal{K}^\dagger|\phi\rangle\big) . \tag{6}$$

This definition of adjoint means that the adjoint works the same way on products as for linear operators. To verify this, we need to check two cases, keeping in mind that the product of two antilinear operators is a linear operator and the product of an antilinear operator and a linear is antilinear:

$$\begin{aligned}
&\langle\psi|(\mathcal{K}\mathcal{L})^\dagger|\phi\rangle = \langle\phi|(\mathcal{K}\mathcal{L})|\psi\rangle^* = (\langle\phi|\mathcal{K})(\mathcal{L}|\psi\rangle) = [(\langle\psi|\mathcal{L}^\dagger)(\mathcal{K}^\dagger|\phi\rangle)]^* = \langle\psi|(\mathcal{L}^\dagger\mathcal{K}^\dagger|\phi\rangle)\;, \\
&\langle\psi|[(\mathcal{K}A)^\dagger|\phi\rangle] = \langle\phi|(\mathcal{K}A|\psi\rangle) = [(\langle\phi|\mathcal{K})(A|\psi\rangle)]^* = (\langle\psi|A^\dagger)(\mathcal{K}^\dagger|\phi\rangle) = \langle\psi|(A^\dagger\mathcal{K}^\dagger|\phi\rangle)\;.
\end{aligned} \tag{7}$$

Thus we have that $(\mathcal{K}\mathcal{L})^\dagger = \mathcal{L}^\dagger\mathcal{K}^\dagger$ and $(\mathcal{K}A)^\dagger = A^\dagger\mathcal{K}^\dagger$. Notice that the linear operators $\mathcal{K}^\dagger\mathcal{K}$ and $\mathcal{K}\mathcal{K}^\dagger$ are Hermitian and, indeed, positive.

An antilinear operator is specified by its "matrix elements" in any orthonormal basis $|e_j\rangle$:

$$\langle e_j|\big(\mathcal{K}|e_k\rangle\big) = \big[\big(\langle e_j|\mathcal{K}\big)|e_k\rangle\big]^* = \langle e_k|\big(\mathcal{K}^\dagger|e_j\rangle\big)\;. \tag{8}$$

Given an orthonormal basis, there is an associated linear operator A that acts the same way as $\mathcal{K}$ on the basis vectors,

$$A|e_j\rangle = \mathcal{K}|e_j\rangle\;, \tag{9}$$

and thus has the same matrix elements in this basis:

$$\langle e_j|A|e_k\rangle = \langle e_j|\big(\mathcal{K}|e_k\rangle\big)\;. \tag{10}$$

The relation between $\mathcal{K}$ and A becomes clear when $\mathcal{K}$ acts on an arbitrary vector

$$\begin{aligned}
\mathcal{K}|\psi\rangle = \mathcal{K}\Bigg(\sum_j c_j|e_j\rangle\Bigg) = \sum_j c_j^*\mathcal{K}|e_j\rangle = \sum_j c_j^* A|e_j\rangle = A\Bigg(\sum_j c_j^*|e_j\rangle\Bigg)& \\
= A|\psi^*\rangle = A\mathcal{C}|\psi\rangle = \mathcal{C}A^*|\psi\rangle\;.&
\end{aligned} \tag{11}$$

Here the complex conjugations on $|\psi\rangle$ and A stand for complex conjugation in the chosen basis $|e_j\rangle$, and $\mathcal{C}$ is the antilinear operator that complex conjugates in this basis. The antilinear operator can be written simply as

$$\mathcal{K} = A\mathcal{C} = \mathcal{C}A^*\;. \tag{12}$$

The matrix elements of $\mathcal{C}$ are

$$\delta_{jk} = \langle e_j|\big(\mathcal{C}|e_k\rangle\big) = \big[\big(\langle e_j|\mathcal{C}\big)|e_k\rangle\big]^* = \langle e_k|\big(\mathcal{C}^\dagger|e_j\rangle\big)\;. \tag{13}$$

We can conclude that

$$\big(\langle e_j|\mathcal{C}\big) = \langle e_j|\;, \tag{14}$$

which means that when acting to the left, $\mathcal{C}$ complex conjugates in the same basis. We can also conclude that

$$\mathcal{C}^\dagger|e_j\rangle = |e_j\rangle\;, \tag{15}$$

from which it follows that $\mathcal{C}^\dagger = \mathcal{C}$. Since it is obvious that $\mathcal{C}^{-1} = \mathcal{C}$, we can summarize the important properties of complex conjugation by

$$\mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^\dagger \; . \tag{16}$$

Notice that

$$\mathcal{K}^\dagger = \mathcal{C}A^\dagger = A^T\mathcal{C} \; , \tag{17}$$

where $A^T \equiv (A^*)^\dagger$ denotes transposition in the chosen basis.

There are two lessons here. The first is that all antilinear operators can be constructed from the linear operators and a single antilinear operator, which can be chosen to be complex conjugation in a particular basis. The second is that antilinear operators turn the basis-independent operations of complex conjugation and transposition into basis-independent operations on antilinear operators.

An *antiunitary operator* $\mathcal{U}$ is an antilinear operator that preserves the absolute value of inner products, i.e.,

$$|\langle\phi|\psi\rangle| = \big|\big(\langle\phi|\mathcal{U}^\dagger\big)\big(\mathcal{U}|\psi\rangle\big)\big| = \big|\langle\phi|\big(\mathcal{U}^\dagger\mathcal{U}|\psi\rangle\big)\big| \tag{18}$$

for all $|\phi\rangle$ and $|\psi\rangle$. Since $\mathcal{C}$ complex conjugates inner products, it is antinunitary. There are two useful characterizations of an antiunitary operator: *the antiunitary of* $\mathcal{U}$ *is equivalent to* (i) $\mathcal{U}^\dagger\mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$ and (ii) $\mathcal{U} = U\mathcal{C} = \mathcal{C}U^*$, *$U$ being unitary, in any basis (and then all bases).* It is obvious that if $\mathcal{U}^\dagger\mathcal{U} = 1$, then $\mathcal{U}$ is antiunitary. Suppose, then, that $\mathcal{U}$ is antiunitary. The definition (18) of antiunitarity, applied to an orthonormal basis, implies that $\mathcal{U}^\dagger\mathcal{U} = e^{i\Lambda}$, where Λ is Hermitian and diagonal in the chosen basis. But, since $\mathcal{U}^\dagger\mathcal{U}$ is positive, we have that $\mathcal{U}^\dagger\mathcal{U} = 1$. The second characterization is trivially equivalent to the first.

Consider now a change of orthonormal basis:

$$|f_j\rangle = V^\dagger|e_j\rangle = \sum_k |e_k\rangle(V^\dagger)_{kj} = \sum_k |e_k\rangle V^*_{jk} \; . \tag{19}$$

The matrix elements of an antilinear operator $\mathcal{K}$ change from one basis to another according to

$$\langle f_j|\big(\mathcal{K}|f_k\rangle\big) = \left(\sum_l V_{jl}\langle e_l|\right)\left[\mathcal{K}\left(\sum_m V^*_{km}|e_m\rangle\right)\right] = \sum_{l,m} V_{jl}\langle e_l|\big(\mathcal{K}|e_m\rangle\big)(V^T)_{mk} \; . \tag{20}$$

We need to say a word about how linear operators transform under antiunitaries. Consider a linear operator A and an antiunitary $\mathcal{U}$:

$$(\langle\psi|\mathcal{U}^\dagger)A(\mathcal{U}|\psi\rangle) = [\langle\psi|(\mathcal{U}^\dagger A\mathcal{U}|\psi\rangle)]^* = \langle\psi|\mathcal{U}^\dagger A\mathcal{U}|\psi\rangle^* = \langle\psi|\mathcal{U}^\dagger A^\dagger\mathcal{U}|\psi\rangle \; . \tag{21}$$

Thus the appropriate antiunitary transformation of a linear operator A is $\mathcal{U}^\dagger A^\dagger\mathcal{U}$. If A is Hermitian, this reduces to the same form as for a unitary transformation, but for nonHermitian operators, it different by having to take the adjoint of A.

II. Polar decompositions

An extremely useful tool is the *polar decomposition* of an antilinear operator $\mathcal{K}$. First diagonalize the positive operator $\mathcal{K}^\dagger\mathcal{K}$:

$$\mathcal{K}^\dagger\mathcal{K} = \sum_j \lambda_j^2 |e_j\rangle\langle e_j| \;, \qquad \lambda_j \ge 0 \;. \tag{22}$$

Relative to the eigenbasis $|e_j\rangle$, $\mathcal{K}$ can be written as

$$\mathcal{K} = A\mathcal{C} = \mathcal{C}A^* \;, \tag{23}$$

where $\mathcal{C}$ is complex conjugation in the eigenbasis. Notice that

$$\mathcal{K}^\dagger\mathcal{K} = \mathcal{C}A^\dagger A\mathcal{C} = A^T A^* \;. \tag{24}$$

Since $\mathcal{K}^\dagger\mathcal{K}$ is real in the eigenbasis, however, we can simplify this to

$$A^\dagger A = \mathcal{C}\mathcal{K}^\dagger\mathcal{K}\mathcal{C} = \mathcal{K}^\dagger\mathcal{K} = A^T A^* = \sum_j \lambda_j^2 |e_j\rangle\langle e_j| \;. \tag{25}$$

The polar-decomposition theorem for A says that there exists a unitary operator U such that

$$A = U\sqrt{A^\dagger A} = \sqrt{AA^\dagger}U = \sum_j \lambda_j |f_j\rangle\langle e_j| \;, \tag{26}$$

where

$$U|e_j\rangle = |f_j\rangle \qquad \Longrightarrow \qquad U^*|e_j\rangle = |f_j^*\rangle \tag{27}$$

and

$$AA^\dagger = UA^\dagger AU^\dagger = \sum_j \lambda_j^2 |f_j\rangle\langle f_j| \;. \tag{28}$$

Notice that $AA^\dagger$ is not real in the eigenbasis:

$$(AA^\dagger)^* = A^*A^T = U^*A^\dagger AU^T = \sum_j \lambda_j^2 |f_j^*\rangle\langle f_j^*| \;. \tag{29}$$

We also have

$$\begin{aligned} A^\dagger &= \sqrt{A^\dagger A}U^\dagger = U^\dagger\sqrt{AA^\dagger} = \sum_j \lambda_j |e_j\rangle\langle f_j| \;, \\ A^* &= U^*\sqrt{A^\dagger A} = \sum_j \lambda_j |f_j^*\rangle\langle e_j| \;, \\ A^T &= \sqrt{A^\dagger A}U^T = \sum_j \lambda_j |e_j\rangle\langle f_j^*| \;. \end{aligned} \tag{30}$$

Using the polar decomposition of A, we can obtain a number of useful forms for the polar decomposition of $\mathcal{K}$, summarized in the following:

$$
\begin{aligned}
\mathcal{K} &= A\mathcal{C} = U\sqrt{\mathcal{K}^\dagger\mathcal{K}}\mathcal{C} = \sqrt{\mathcal{K}\mathcal{K}^\dagger}U\mathcal{C} = \left(\sum_j \lambda_j|f_j\rangle\langle e_j|\right)\mathcal{C}\;, \\
\mathcal{K} &= \mathcal{C}A^* = \mathcal{C}U^*\sqrt{\mathcal{K}^\dagger\mathcal{K}} = \mathcal{C}\left(\sum_j \lambda_j|f_j^*\rangle\langle e_j|\right)\;.
\end{aligned} \tag{31}
$$

Other useful relations include

$$
\begin{aligned}
\mathcal{K}^\dagger\mathcal{K} &= A^\dagger A = A^T A^* = U^\dagger\mathcal{K}\mathcal{K}^\dagger U = \sum_j \lambda_j^2|e_j\rangle\langle e_j|\;, \\
\mathcal{K}\mathcal{K}^\dagger &= AA^\dagger = U\mathcal{K}^\dagger\mathcal{K}U^\dagger = \sum_j \lambda_j^2|f_j\rangle\langle f_j|\;.
\end{aligned} \tag{32}
$$

Notice that $U^\dagger\mathcal{K}\mathcal{K}^\dagger U$, not $\mathcal{K}\mathcal{K}^\dagger$, is real in the eigenbasis.

III. Symmetric antilinear operators

We are now ready to consider *antilinear Hermitian* (or *symmetric*) antilinear operators, i.e., antilinear operators satisfying $\mathcal{K} = \mathcal{K}^\dagger$. *That $\mathcal{K}$ is antilinear Hermitian is equivalent to each of the following:*

(i) *$\mathcal{K}$ is symmetric, i.e., $\langle\phi|\big(\mathcal{K}|\psi\rangle\big) = \langle\psi|\big(\mathcal{K}|\phi\rangle\big)$ for all $|\phi\rangle$ and $|\psi\rangle$.*

(ii) *$\mathcal{K}$ is symmetric in any one orthonormal basis (and then all orthonormal bases).*

(iii) *$\mathcal{K}$ can be diagonalized in an orthonormal basis; i.e., there exists an orthonormal basis $|e_j\rangle$ such that $\mathcal{K}|e_j\rangle = \lambda_j|e_j\rangle$. The "eigenvalues" λ_j can be made real and positive.*

The first two of these are trivial consequences of the definition of the adjoint. Before considering the third, notice that $\mathcal{K}^2|e_j\rangle = \mathcal{K}^\dagger\mathcal{K}|e_j\rangle = |\lambda_j|^2|e_j\rangle$. Moreover, notice that the reason that the "eigenvalues" can be made real and positive is that under a rephasing of the eigenvectors, i.e., $|e_j'\rangle = e^{i\delta_j}|e_j\rangle$, the "eigenvalues" undergo a phase change,

$$
\mathcal{K}|e_j'\rangle = \mathcal{K}\big(e^{i\delta_j}|e_j\rangle\big) = e^{-i\delta_j}\mathcal{K}|e_j\rangle = e^{-i\delta_j}\lambda_j|e_j\rangle = \lambda_j e^{-2i\delta_j}|e_j'\rangle\;. \tag{33}
$$

The real and positive "eigenvalues" λ_j can be characterized uniquely as the square roots of the eigenvalues of $\mathcal{K}^\dagger\mathcal{K} = \mathcal{K}^2$.

Now consider item (iii). One direction is trivial: if there is such a basis, then $\mathcal{K}$, being diagonal in that basis, is also symmetric in that basis, and hence $\mathcal{K} = \mathcal{K}^\dagger$. So now suppose that $\mathcal{K} = \mathcal{K}^\dagger$, and show that the desired basis exists. We proceed by using the polar decomposition of $\mathcal{K}$, being careful now to label explicitly any degeneracies:

$$
\begin{aligned}
\mathcal{K} &= \left(\sum_\alpha \lambda_\alpha\left(\sum_j |f_{\alpha j}\rangle\langle e_{\alpha j}|\right)\right)\mathcal{C}\;, &\qquad \mathcal{K}^2 &= \mathcal{K}^\dagger\mathcal{K} = \sum_\alpha \lambda_\alpha\left(\sum_j |e_{\alpha j}\rangle\langle e_{\alpha j}|\right)\;, \\
\mathcal{K} = \mathcal{K}^\dagger &= \left(\sum_\alpha \lambda_\alpha\left(\sum_j |e_{\alpha j}\rangle\langle f_{\alpha j}^*|\right)\right)\mathcal{C}\;, &\qquad \mathcal{K}^2 &= \mathcal{K}\mathcal{K}^\dagger = \sum_\alpha \lambda_\alpha\left(\sum_j |f_{\alpha j}\rangle\langle f_{\alpha j}|\right)\;.
\end{aligned} \tag{34}
$$

The expressions on the right imply that within each eigensubspace of $\mathcal{K}^2$, labeled by an index α, we have

$$|f_{\alpha j}\rangle = \sum_k |e_{\alpha k}\rangle V_{kj}^{(\alpha)} \ , \tag{35}$$

where $V_{kj}^{(\alpha)}$ is a unitary matrix. Notice that for the null subspace ($\lambda_\alpha = 0$), we can assume that $V_{jk}^{(\alpha)}$ is the unit matrix, i.e., $|f_{\alpha j}\rangle = |e_{\alpha j}\rangle$. This being the result we want to get to (for this subspace), we can afford to assume that $\lambda_\alpha \neq 0$ in the following. Plugging Eq. (35) into the expressions on the left in Eq. (34) gives

$$\mathcal{K} = \left(\sum_\alpha \lambda_\alpha \left(\sum_{j,k} V_{jk}^{(\alpha)} |e_{\alpha j}\rangle\langle e_{\alpha k}|\right)\right)\mathcal{C} = \left(\sum_\alpha \lambda_\alpha \left(\sum_{j,k} V_{kj}^{(\alpha)} |e_{\alpha j}\rangle\langle e_{\alpha k}|\right)\right)\mathcal{C} \ , \tag{36}$$

thus implying that $V_{jk}^{(\alpha)} = V_{kj}^{(\alpha)}$ is symmetric. A symmetric unitary matrix can be diagonalized by an orthogonal transformation:

$$V_{jk}^{(\alpha)} = \sum_l O_{jl}^{(\alpha)} O_{kl}^{(\alpha)} e^{i\phi_{\alpha l}} \ . \tag{37}$$

(To see this, write $V = e^{iH}$, where $H = H^\dagger$ is Hermitian. The symmetry of V implies that $H = H^T$ is also symmetric and, hence, that $H = H^*$ is real. As a real, symmetric matrix, H—and, hence, V—can be diagonalized by an orthogonal transformation. Another way to see this, which we use below in the antisymmetric case, is to write $V = A + iB$, where $A = (V + V^\dagger)/2 = (V + V^*)/2 = A^* = A^T$ and $B = -i(V - V^\dagger)/2 = -i(V - V^*)/2 = B^* = B^T$ are real and symmetric. The unitarity of V gives $I = V^\dagger V = A^2 + B^2 + i[A, B]$, which implies that $[A, B] = 0$. Thus A and B—and, hence, V—can be diagonalized simultaneously by an orthogonal transformation.)

Now define a new orthonormal basis by defining new vectors within each $\lambda_\alpha \neq 0$ eigensubspace:

$$|e'_{\alpha l}\rangle = \sum_j |e_{\alpha j}\rangle O_{jl}^{(\alpha)} \ . \tag{38}$$

For the null subspace, simply define $|e'_{\alpha j}\rangle = |e_{\alpha j}\rangle$. Since the new basis is related to the old by a (real) orthogonal transformation, complex conjugation in the two bases is the same, i.e., $\mathcal{C} = \mathcal{C}'$. Thus we have

$$\mathcal{K} = \left(\sum_\alpha \lambda_\alpha \left(\sum_l e^{i\phi_{\alpha l}} |e'_{\alpha l}\rangle\langle e'_{\alpha l}|\right)\right)\mathcal{C}' = \left(\sum_j \lambda_j e^{i\phi_j} |e'_j\rangle\langle e'_j|\right)\mathcal{C}' = \mathcal{C}' \sum_j \lambda_j e^{-i\phi_j} |e'_j\rangle\langle e'_j| \ , \tag{39}$$

where in the second expression, we abandon explicit labeling of the degeneracies. We now have $\mathcal{K}$ in diagonal form with $\mathcal{K}|e'_j\rangle = \lambda_j e^{i\phi_j}|e'_j\rangle$. To bring $\mathcal{K}$ into the standard form that has real, positive "eigenvalues," we can rephase the eigenvectors appropriately, i.e., $|e''_j\rangle = e^{i\phi_j/2}|e'_j\rangle$,

$$\begin{aligned}
\mathcal{K} &= \mathcal{C}''(\mathcal{C}''\mathcal{C}')\left(\sum_j \lambda_j e^{-i\phi_j}|e'_j\rangle\langle e'_j|\right) \\
&= \mathcal{C}''\left(\sum_j \lambda_j e^{-i\phi_j}\underbrace{\big(\mathcal{C}''\mathcal{C}'|e'_j\rangle\big)}_{=e^{i\phi_j}|e'_j\rangle}\langle e'_j|\right) \\
&= \mathcal{C}''\left(\sum_j \lambda_j |e'_j\rangle\langle e'_j|\right) \\
&= \mathcal{C}''\left(\sum_j \lambda_j |e''_j\rangle\langle e''_j|\right) \\
&= \left(\sum_j \lambda_j |e''_j\rangle\langle e''_j|\right)\mathcal{C}'' \ .
\end{aligned} \tag{40}$$

A trivial consequence of our general results for symmetric antilinear operators is that *an antiunitary operator $\mathcal{U}$ is diagonalizable in an orthonormal basis if and only if $\mathcal{U} = \mathcal{U}^\dagger$ (or $\mathcal{U}^2 = 1$), which means that $\mathcal{U}$ is complex conjugation in that basis.*

It is worth emphasizing the freedom that is available in the eigenvectors of $\mathcal{K} = \mathcal{K}^\dagger$. For this purpose, we drop the primes in the last form of Eq. (40) and again label degeneracies, writing

$$\mathcal{K} = \left(\sum_\alpha \lambda_\alpha\left(\sum_j |e_{\alpha j}\rangle\langle e_{\alpha j}|\right)\right)\mathcal{C} \qquad \Longleftrightarrow \qquad \langle e_{\alpha j}|\big(\mathcal{K}|e_{\beta k}\rangle\big) = \lambda_\alpha \delta_{\alpha\beta}\delta_{jk} \ . \tag{41}$$

Allowed basis changes are restricted to unitary transformations within each eigensubspace,

$$|f_{\alpha j}\rangle = \sum_k |e_{\alpha k}\rangle V^{(\alpha)*}_{jk} \ , \tag{42}$$

which gives

$$\langle f_{\alpha j}|\big(\mathcal{K}|f_{\beta k}\rangle\big) = \lambda_\alpha \delta_{\alpha\beta}\big(V^{(\alpha)}V^{(\alpha)T}\big)_{jk} \ . \tag{43}$$

In the null subspace ($\lambda_\alpha = 0$), any unitary transformation is allowed, but to maintain a diagonal form in the nonnull subspaces ($\lambda_\alpha \neq 0$) requires that the unitary matrix $\big(V^{(\alpha)}V^{(\alpha)T}\big)$ be diagonal, i.e.,

$$\big(V^{(\alpha)}V^{(\alpha)T}\big)_{jk} = e^{i\phi_j}\delta_{jk} \ . \tag{44}$$

Defining a unitary matrix $O^{(\alpha)}$ by

$$O_{jk}^{(\alpha)} = e^{-i\phi_j/2} V_{jk}^{(\alpha)} \;, \tag{45}$$

we see that

$$\big(O^{(\alpha)} O^{(\alpha)T}\big)_{jk} = e^{-i(\phi_j+\phi_k)/2} \big(V^{(\alpha)} V^{(\alpha)T}\big)_{jk} = \delta_{jk} \;, \tag{46}$$

which means that $O^{(\alpha)}$ is a (real) orthogonal matrix.

We can summarize the freedom in the eigenvectors as follows. Starting from a set of eigenvectors that put $\mathcal{K}$ in the standard form that has real, positive "eigenvalues," we can transform to new eigenvectors in each eigensubpace,

$$|f_{\alpha j}\rangle = e^{-i\phi_j/2} \sum_k |e_{\alpha k}\rangle O_{jk}^{(\alpha)} \;. \tag{47}$$

In the null subspace, any unitary transformation is allowed. In the nonnull subspaces, these new eigenvectors maintain the diagonal form of $\mathcal{K}$ provided that $O_{jk}^{(\alpha)}$ is an orthogonal matrix. The general transformation is thus an orthogonal transformation followed by rephasings of the eigenvectors. If we wish to retain real, positive "eigenvalues," then we are restricted to orthogonal transformations without the phases. In each nonnull eigensubspace of $\mathcal{K}^2$, $\mathcal{K}$ picks out a unique real subspace of the same dimension, which puts $\mathcal{K}$ in the standard form with real, positive "eigenvalues." For nondegenerate eigensubspaces, this real subspace is simply a unique (up to a sign change) phase choice for the eigenvector. The proof of item (iii) shows that to diagonalize a symmetric antilinear operator $\mathcal{K}$, one first diagonalizes the linear operator $\mathcal{K}^2$ and then in each nonnull eigensubspace of $\mathcal{K}^2$, one finds the unique real subspace that puts $\mathcal{K}$ in the standard form with real, positive eigenvalues.

I still want to include the proof of minimal "trace" for symmetric antilinears.

IV. Antisymmetric antilinear operators

We turn now to *antilinear antiHermitian* (or *antisymmetric*) antilinear operators, i.e., those satisfying $\mathcal{K} = -\mathcal{K}^\dagger$. Much of the discussion is nearly identical to that for symmetric antilinear operators, but we can afford just to repeat it, there being little point in trying to save space. It might be useful to consult the companion report entitled *Antisymmetric operators on a real vector space* before reading this section.

That $\mathcal{K}$ is antilinear antiHermitian is equivalent to each of the following:

(i) *$\mathcal{K}$ is antisymmetric, i.e., $\langle\phi|\big(\mathcal{K}|\psi\rangle\big) = -\langle\psi|\big(\mathcal{K}|\phi\rangle\big)$ for all $|\phi\rangle$ and $|\psi\rangle$.*

(ii) *$\mathcal{K}$ is antisymmetric in any one orthonormal basis (and then all orthonormal bases).*

(iii) *There exists an orthonormal basis $|e_{j\sigma}\rangle$ where for each j, $\sigma = \pm 1$ (pairs of basis vectors) or $\sigma = 0$ (single basis vectors), such that $\mathcal{K}|e_{j\sigma}\rangle = -\sigma\lambda_j|e_{j,-\sigma}\rangle$; i.e., $\mathcal{K}$ has matrix elements*

$$\langle e_{j\sigma}|\big(\mathcal{K}|e_{k\tau}\rangle\big) = \lambda_j \delta_{jk} \underbrace{\sigma\delta_{\sigma,-\tau}}_{=\,\epsilon_{\sigma\tau}} \;.$$

The quantities λ_j can be made real and positive.

The first two of these are trivial consequences of the definition of the adjoint. Before considering the third, notice that $\mathcal{K}^2|e_{j\sigma}\rangle = -\mathcal{K}^\dagger\mathcal{K}|e_{j\sigma}\rangle = -\sigma^2|\lambda_j|^2|e_{j\sigma}\rangle$. Moreover, notice that the reason that the λ_j's can be made real and positive is that under a rephasing of the basis vectors, i.e., $|e'_{j\sigma}\rangle = e^{i\delta_{j\sigma}}|e_{j\sigma}\rangle$, the λ_j's undergo a phase change,

$$\begin{aligned}\mathcal{K}|e'_{j\sigma}\rangle &= \mathcal{K}\big(e^{i\delta_{j\sigma}}|e_{j\sigma}\rangle\big)\\ &= e^{-i\delta_{j\sigma}}\mathcal{K}|e_{j\sigma}\rangle\\ &= e^{-i\delta_{j\sigma}}\big(-\sigma\lambda_j|e_{j,-\sigma}\rangle\big)\\ &= -\sigma\lambda_j e^{-i(\delta_{j\sigma}+\delta_{j,-\sigma})}|e'_{j,-\sigma}\rangle\ .\end{aligned}\tag{48}$$

For a single basis vector, the rephasing is irrelevant because $\mathcal{K}|e_{j0}\rangle = 0$; for a pair of eigenvectors, λ_j is rephased by $e^{-i(\delta_{j,+1}+\delta_{j,-1})}$. The real and positive quantities λ_j can be characterized uniquely as the square roots of the eigenvalues of $\mathcal{K}^\dagger\mathcal{K} = -\mathcal{K}^2$.

Now consider item (iii). One direction is trivial: if there is such a basis, then $\mathcal{K}$ is clearly antisymmetric in that basis and, hence, $\mathcal{K} = -\mathcal{K}^\dagger$. So now suppose that $\mathcal{K} = -\mathcal{K}^\dagger$, and show that the desired basis exists. We proceed by using the polar decomposition of $\mathcal{K}$, being careful now to label explicitly any degeneracies:

$$\begin{aligned}\mathcal{K} &= \left(\sum_\alpha \lambda_\alpha\left(\sum_j |f_{\alpha j}\rangle\langle e_{\alpha j}|\right)\right)\mathcal{C}\ , &\quad -\mathcal{K}^2 = \mathcal{K}^\dagger\mathcal{K} &= \sum_\alpha \lambda_\alpha\left(\sum_j |e_{\alpha j}\rangle\langle e_{\alpha j}|\right)\ ,\\ -\mathcal{K} = \mathcal{K}^\dagger &= \left(\sum_\alpha \lambda_\alpha\left(\sum_j |e_{\alpha j}\rangle\langle f^*_{\alpha j}|\right)\right)\mathcal{C}\ , &\quad -\mathcal{K}^2 = \mathcal{K}\mathcal{K}^\dagger &= \sum_\alpha \lambda_\alpha\left(\sum_j |f_{\alpha j}\rangle\langle f_{\alpha j}|\right)\ .\end{aligned}\tag{49}$$

The expressions on the right imply that within each eigensubspace of $\mathcal{K}^2$, labeled by an index α, we have

$$|f_{\alpha j}\rangle = \sum_k |e_{\alpha k}\rangle V^{(\alpha)}_{kj}\ ,\tag{50}$$

where $V^{(\alpha)}_{kj}$ is a unitary matrix. Notice that for the null subspace ($\lambda_\alpha = 0$), we can assume that $V^{(\alpha)}_{jk}$ is the unit matrix, i.e., $|f_{\alpha j}\rangle = |e_{\alpha j}\rangle$. These null-subspace basis vectors serve as the single basis vectors ($\sigma = 0$) in the ultimate basis we seek. Having dealt with the $\lambda_\alpha = 0$ case, we can afford to assume that we are dealing with the $\lambda_\alpha \neq 0$ subspaces in the following. Plugging Eq. (50) into the expressions on the left in Eq. (49) gives

$$\mathcal{K} = \left(\sum_\alpha \lambda_\alpha\left(\sum_{j,k} V^{(\alpha)}_{jk}|e_{\alpha j}\rangle\langle e_{\alpha k}|\right)\right)\mathcal{C} = \left(\sum_\alpha \lambda_\alpha\left(-\sum_{j,k} V^{(\alpha)}_{kj}|e_{\alpha j}\rangle\langle e_{\alpha k}|\right)\right)\mathcal{C}\ ,\tag{51}$$

thus implying that $V^{(\alpha)}_{jk} = -V^{(\alpha)}_{kj}$ is antisymmetric. An antisymmetric unitary matrix can be brought into the following form by an orthogonal transformation:

$$V^{(\alpha)}_{jk} = \sum_{l,\sigma,\tau} O^{(\alpha)}_{j,l\sigma}O^{(\alpha)}_{k,l\tau}e^{i\phi_{\alpha l}}\epsilon_{\sigma\tau}\ .\tag{52}$$

(To see this, write $V = iA+B$, where $A = -i(V+V^\dagger)/2 = -i(V-V^*)/2 = A^* = -A^T$ and $B = (V - V^\dagger)/2 = (V + V^*)/2 = B^* = -B^T$ are real and antisymmetric. The unitarity of V gives $I = V^\dagger V = A^2 + B^2 - i[A, B]$, which implies that $[A, B] = 0$. Thus A and B—and, hence, V—can be simultaneously brought into the desired form by an orthogonal transformation, i.e.,

$$\left.\begin{aligned} A_{jk} &= \sum_{l,\sigma,\tau} O_{j,l\sigma} O_{k,l\tau} \mu_l \epsilon_{\sigma\tau} \\ B_{jk} &= \sum_{l,\sigma,\tau} O_{j,l\sigma} O_{k,l\tau} \nu_l \epsilon_{\sigma\tau} \end{aligned}\right. \qquad \Longrightarrow \qquad V_{jk} = \sum_{l,\sigma,\tau} O_{j,l\sigma} O_{k,l\tau} (i\mu_l + \nu_l) \epsilon_{\sigma\tau} \ .$$

The unitarity of V guarantees, first, that the $\sigma = 0$ case does not occur and, second, that $i\mu_l + \nu_l = e^{i\phi_l}$ is a phase, thus bringing V_{jk} into the desired form.)

Now define a new orthonormal basis by defining new vectors within each $\lambda_\alpha \neq 0$ eigensubspace:

$$|e'_{\alpha,l\sigma}\rangle = \sum_j |e_{\alpha j}\rangle O^{(\alpha)}_{j,l\sigma} \ . \tag{53}$$

For the null subspace, simply define $|e'_{\alpha,j0}\rangle = |e_{\alpha j}\rangle$ ($\sigma = 0$). Since the new basis is related to the old by a (real) orthogonal transformation, complex conjugation in the two bases is the same, i.e., $\mathcal{C} = \mathcal{C}'$. Thus we have

$$\begin{aligned} \mathcal{K} &= \left(\sum_\alpha \lambda_\alpha \left(\sum_{l,\sigma,\tau} e^{i\phi_{\alpha l}} \epsilon_{\sigma\tau} |e'_{\alpha,l\sigma}\rangle\langle e'_{\alpha,l\tau}| \right) \right) \mathcal{C}' \\ &= \left(\sum_{j,\sigma,\tau} \lambda_j e^{i\phi_j} \epsilon_{\sigma\tau} |e'_{j\sigma}\rangle\langle e'_{j\tau}| \right) \mathcal{C}' \\ &= \mathcal{C}' \sum_{j,\sigma,\tau} \lambda_j e^{-i\phi_j} \epsilon_{\sigma\tau} |e'_{j\sigma}\rangle\langle e'_{j\tau}| \ , \end{aligned} \tag{54}$$

where in the second expression, we abandon explicit labeling of the degeneracies. We now have $\mathcal{K}$ in the desired form with

$$\mathcal{K}|e'_{j\sigma}\rangle = \sum_\tau \lambda_j e^{i\phi_j} \epsilon_{\tau\sigma} |e'_{j\tau}\rangle = -\sigma \lambda_j e^{i\phi_j} |e'_{j,-\sigma}\rangle \ .$$

To bring $\mathcal{K}$ into the standard form that has real, positive λ_j's, we can rephase the eigenvectors appropriately, i.e., $|e''_{j\sigma}\rangle = e^{i(\phi_j+\sigma\theta_j)/2)}|e'_{j\sigma}\rangle$. This final basis change brings $\mathcal{K}$ into the form

$$\begin{aligned}
\mathcal{K} &= \mathcal{C}''(\mathcal{C}''\mathcal{C}')\left(\sum_{j,\sigma,\tau}\lambda_j e^{-i\phi_j}\epsilon_{\sigma\tau}|e'_{j\sigma}\rangle\langle e'_{j\tau}|\right) \\
&= \mathcal{C}''\left(\sum_{j,\sigma,\tau}\lambda_j e^{-i\phi_j}\epsilon_{\sigma\tau}\underbrace{\big(\mathcal{C}''\mathcal{C}'|e'_{j\sigma}\rangle\big)}_{=e^{i(\phi_j+\sigma\theta_j)/2}|e''_{j\sigma}\rangle}\langle e'_{j\tau}|\right) \\
&= \mathcal{C}''\left(\sum_{j,\sigma,\tau}\lambda_j\epsilon_{\sigma\tau}e^{i(\sigma+\tau)\theta_j/2}|e''_{j\sigma}\rangle\langle e''_{j\tau}|\right) \\
&= \mathcal{C}''\left(\sum_{j,\sigma,\tau}\lambda_j\epsilon_{\sigma\tau}|e''_{j\sigma}\rangle\langle e''_{j\tau}|\right) \\
&= \left(\sum_{j,\sigma,\tau}\lambda_j\epsilon_{\sigma\tau}|e''_{j\sigma}\rangle\langle e''_{j\tau}|\right)\mathcal{C}'' \ .
\end{aligned} \tag{55}$$

A trivial consequence of our general results for antisymmetric antilinear operators is that *an antiunitary operator $\mathcal{U}$ can be brought to the form of item (iii) in an orthonormal basis if and only if $\mathcal{U} = -\mathcal{U}^\dagger$ (or $\mathcal{U}^2 = -1$).*

It is worth emphasizing the freedom that is available in the eigenvectors of $\mathcal{K} = -\mathcal{K}^\dagger$. For this purpose, we drop the primes in the last form of Eq. (55) and again label degeneracies, writing

$$\mathcal{K} = \left(\sum_\alpha \lambda_\alpha\left(\sum_{j,\sigma,\tau}\epsilon_{\sigma\tau}|e_{\alpha,j\sigma}\rangle\langle e_{\alpha,j\tau}|\right)\right)\mathcal{C} \qquad \Longleftrightarrow \qquad \langle e_{\alpha,j\sigma}|\big(\mathcal{K}|e_{\beta,k\tau}\rangle\big) = \lambda_\alpha\delta_{\alpha\beta}\delta_{jk}\epsilon_{\sigma\tau} \ . \tag{56}$$

Allowed basis changes are restricted to unitary transformations within each eigensubspace,

$$|f_{\alpha,j\sigma}\rangle = \sum_{k,\tau}|e_{\alpha,k\tau}\rangle V^{(\alpha)*}_{j\sigma,k\tau} \ , \tag{57}$$

which gives

$$\langle f_{\alpha,j\sigma}|\big(\mathcal{K}|f_{\beta,k\tau}\rangle\big) = \lambda_\alpha\delta_{\alpha\beta}\sum_{l,\mu,\nu}V^{(\alpha)}_{j\sigma,l\mu}\epsilon_{\mu\nu}V^{(\alpha)}_{k\tau,l\nu} \ . \tag{58}$$

In the null subspace ($\lambda_\alpha = 0$), any unitary transformation is allowed, but to maintain the desired matrix elements in the nonnull subspaces ($\lambda_\alpha \neq 0$) requires that

$$\sum_{l,\mu,\nu}V^{(\alpha)}_{j\sigma,l\mu}\epsilon_{\mu\nu}V^{(\alpha)}_{k\tau,l\nu} = e^{i\phi_j}\delta_{jk}\epsilon_{\sigma\tau} \ . \tag{59}$$

Defining a unitary matrix $M^{(\alpha)}$ by

$$M^{(\alpha)}_{j\sigma,k\tau} = e^{-i\phi_j/2} V^{(\alpha)}_{j\sigma,k\tau} \ , \tag{60}$$

we see that

$$\sum_{l,\mu,\nu} M^{(\alpha)}_{j\sigma,l\mu} \epsilon_{\mu\nu} M^{(\alpha)}_{k\tau,l\nu} = \delta_{jk} \epsilon_{\sigma\tau} \ . \tag{61}$$

We need a little more efficient notation to appreciate the consequences of this requirement. Define the symplectic form

$$\epsilon_{j\sigma,k\tau} = \delta_{jk} \epsilon_{\sigma\tau} \ , \tag{62}$$

and let ϵ stand for this antisymmetric matrix, which satisfies $\epsilon = \epsilon^* = -\epsilon^T = -\epsilon^\dagger$ and $\epsilon^2 = -1$. Equation (61), written in matrix notation, becomes

$$M^{(\alpha)} \epsilon M^{(\alpha)T} = \epsilon \ . \tag{63}$$

This requirement means that $M^{(\alpha)}$ is a complex symplectic matrix, in addition to being unitary. We can put the requirement (63) in an explicit form by first taking the adjoint, $M^{(\alpha)*} \epsilon M^{(\alpha)\dagger} = \epsilon$, multiplying by ϵ on the right, $\epsilon M^{(\alpha)*} \epsilon M^{(\alpha)\dagger} = -1$, and and then comparing with $M^{(\alpha)} M^{(\alpha)\dagger} = 1$ to get $M^{(\alpha)} = -\epsilon M^{(\alpha)*} \epsilon$. Writing out this last equation in index form gives

$$M^{(\alpha)}_{j\sigma,k\tau} = \sum_{\mu,\nu} \epsilon_{\sigma\mu} \epsilon_{\tau\nu} M^{(\alpha)*}_{j\mu,k\nu} = \sigma\tau M^{(\alpha)*}_{j,-\sigma;k,-\tau} \ , \tag{64}$$

which gives the explicit relations

$$M^{(\alpha)}_{j,+1;k,+1} = M^{(\alpha)*}_{j,-1;k,-1} \ , \qquad M^{(\alpha)}_{j,+1;k,-1} = -M^{(\alpha)*}_{j,-1;k,+1} \ . \tag{65}$$

We can summarize the freedom in the basis vectors as follows. Starting from a set of orthonormal vectors that put $\mathcal{K}$ in the standard form that has real, positive λ_j's, we can transform to new eigenvectors in each eigensubpace,

$$|f_{\alpha,j\sigma}\rangle = e^{-i\phi_j/2} \sum_{k,\tau} |e_{\alpha,k\tau}\rangle M^{(\alpha)*}_{j\sigma,k\tau} \ . \tag{66}$$

In the null subspace, any unitary transformation is allowed. In the nonnull subspaces, these new orthonormal vectors maintain the diagonal form of $\mathcal{K}$ provided that $M^{(\alpha)}_{j\sigma,k\tau}$ is a complex symplectic matrix. The general transformation is thus a complex symplectic transformation followed by rephasings of the basis vectors. If we wish to retain real, positive λ_j's, then we are restricted to complex symplectic transformations without the phases. For nondegenerate (two-dimensional) subspaces, the matrix M can be any unit-determinant unitary matrix

$$M = ||M_{\sigma\tau}|| = \begin{pmatrix} e^{i\theta}\cos\theta & e^{i\chi}\sin\theta \\ -e^{-i\chi}\sin\theta & e^{-i\theta}\cos\theta \end{pmatrix} \ . \tag{67}$$