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Subject: **Symmetric informationally complete POVMs**
1999 September 9; modified 2002 June 18 to include material on $\mathcal{G}$

In a D-dimensional Hilbert space, a symmetric informationally complete POVM is a set of one-dimensional projectors $\Pi_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$, $\alpha = 1, \ldots, D^2$, satisfying

$$I = \frac{1}{D}\sum_\alpha \Pi_\alpha = \frac{1}{D}\sum_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \tag{1}$$

(the I/D is dictated by taking the trace) and

$$\mathrm{tr}(\Pi_\alpha\Pi_\beta) = |\langle\psi_\alpha|\psi_\beta\rangle|^2 = \mu^2 \qquad \text{for } \alpha \neq \beta. \tag{2}$$

We can easily derive the value of μ by squaring Eq. (1) and taking the trace of the result

$$D = \mathrm{tr}(I^2) = \frac{1}{D^2}\sum_{\alpha,\beta}\mathrm{tr}(\Pi_\alpha\Pi_\beta) = \frac{1}{D^2}\big(D^2 + \mu^2 D^2(D^2-1)\big)\ . \tag{3}$$

This gives

$$\mu^2 = \frac{1}{D+1}\ . \tag{4}$$

It is easy to show that the projectors Π_α are linearly independent. [This is required for an informationally complete POVM; in this symmetric case, we need not assume it since it follows from Eqs. (1) and (2).] Suppose that

$$0 = \sum_\alpha c_\alpha \Pi_\alpha\ . \tag{5}$$

This implies that for each β,

$$0 = \sum_\alpha c_\alpha\langle\psi_\beta|\Pi_\alpha|\psi_\beta\rangle = c_\beta + \mu^2\sum_{\alpha\neq\beta} c_\alpha = (1-\mu^2)c_\beta + \mu^2\sum_\alpha c_\alpha = \frac{1}{D+1}\left(Dc_\beta + \sum_\alpha c_\alpha\right)\ , \tag{6}$$

and, hence, that

$$c_\beta = -\frac{1}{D}\sum_\alpha c_\alpha \equiv A \tag{7}$$

is independent of β. But then Eq. (7) shows that $A = 0$ and thus that all the c_β's are zero. We conclude that the projectors Π_α are linearly independent, and there being D^2 of them, they make up an operator basis.

Another property that we get for free is the following. Consider the superoperator

$$\mathcal{G} = \sum_\alpha \Pi_\alpha \odot \Pi_\alpha = \sum_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \odot |\psi_\alpha\rangle\langle\psi_\alpha|\ . \tag{8}$$

Applying this operator to one of the projectors, we get

$$
\begin{aligned}
\mathcal{G}(\Pi_\beta) &= \sum_\alpha \Pi_\alpha |\langle\psi_\alpha|\psi_\beta\rangle|^2 \\
&= \Pi_\beta + \frac{1}{D+1}\sum_{\alpha\neq\beta}\Pi_\alpha \\
&= \frac{D}{D+1}\Pi_\beta + \frac{1}{D+1}\sum_\alpha \Pi_\alpha \\
&= \frac{D}{D+1}\big(\Pi_\beta + I\big)\;.
\end{aligned}
\tag{9}
$$

Since the projectors Π_β are a complete set of operators, this implies that $\mathcal{G}$ is the special superoperator

$$
\mathcal{G} = \frac{D}{D+1}\big(\mathcal{I} + \mathbf{I}\big)\;,
\tag{10}
$$

which is specified up to a scale factor by the properties (i) $\mathcal{G} = \mathcal{G}^\dagger = \mathcal{G}^\times = \mathcal{G}^\#$ and (ii) $\mathcal{G}$ commutes with all unitaries, i.e., $U^\dagger \odot U \circ \mathcal{G} \circ U \odot U^\dagger = \mathcal{G}$. The scale factor comes from $\mathrm{Tr}(\mathcal{G}) = D^2$.

We can now find the (left-right) inverse of $\mathcal{G}$ in the following way. Use $\mathbf{I} = I \odot I/D + \mathcal{T} = \mathcal{I}/D + \mathcal{T}$, where $\mathcal{T}$ is the (left-right) projector onto trace-free operators, to write $\mathcal{G}$ in terms of its (left-right) eigendecomposition

$$
\mathcal{G} = \mathcal{I} + \frac{D}{D+1}\mathcal{T} = D\frac{I\odot I}{D} + \frac{D}{D+1}\mathcal{T}\;,
\tag{11}
$$

from which it easy to find the inverse:

$$
\mathcal{G}^{-1} = \frac{1}{D}\frac{I\odot I}{D} + \frac{D+1}{D}\mathcal{T} = \frac{\mathcal{I}}{D^2} + \frac{D+1}{D}\mathcal{T} = \frac{1}{D}\Big((D+1)\mathbf{I} - \mathcal{I}\Big)\;.
\tag{12}
$$

This allows us to write

$$
Q_\alpha \equiv \mathcal{G}^{-1}|\Pi_\alpha) = \frac{1}{D}\big((D+1)\Pi_\alpha - I\big)
\tag{13}
$$

and

$$
\mathbf{I} = \mathcal{G}^{-1}\mathcal{G} = \sum_\alpha |Q_\alpha)(\Pi_\alpha| = \frac{1}{D}\sum_\alpha\big((D+1)\Pi_\alpha\odot\Pi_\alpha - I\odot\Pi_\alpha\big)\;.
\tag{14}
$$

If you didn't like the derivation of this equation, it can easily be checked as follows:

$$
\frac{1}{D}\sum_\alpha\big((D+1)\Pi_\alpha\odot\Pi_\alpha - I\odot\Pi_\alpha\big) = \frac{D+1}{D}\mathcal{G} - I\odot I = \mathcal{I} + \mathbf{I} - \mathcal{I} = \mathbf{I}\;.
\tag{15}
$$

Equation (14) is the result we are looking for. It allows us to write any state ρ in the form

$$\begin{aligned}
\rho = \mathbf{I}|\rho) &= \sum_\alpha \big((D+1)\Pi_\alpha - I\big)\frac{(\Pi_\alpha|\rho)}{D} \\
&= \sum_\alpha \big((D+1)\Pi_\alpha - I\big)\frac{\mathrm{tr}(\Pi_\alpha\rho)}{D} \\
&= \sum_\alpha p_\alpha\big((D+1)\Pi_\alpha - I\big) \\
&= -I + (D+1)\sum_\alpha p_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| \\
&= \sum_\alpha \left((D+1)p_\alpha - \frac{1}{D}\right)|\psi_\alpha\rangle\langle\psi_\alpha| \, ,
\end{aligned} \tag{16}$$

where $p_\alpha = \mathrm{tr}(\Pi_\alpha\rho)/D$ is the probability to find result α. Putting it slightly differently, we can also write

$$\frac{\rho + I}{D+1} = \sum_\alpha p_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| \, . \tag{17}$$

Notice that

$$q_\alpha = (Q_\alpha|\rho) = \mathrm{tr}(Q_\alpha\rho) = (D+1)p_\alpha - \frac{1}{D} \tag{18}$$

is a normalized quasidistribution, since it is generally takes on negative values. Any informationally complete POVM generates measurement probabilities p_α and a unique quasidistribution q_α that comes from expanding the state in terms of the projectors in the POVM. Symmetric informational complete POVMs have a uniquely simple relation between the measurement probabilities and the quasidistribution, which allows one to write down immediately the quasidistribution expansion of an arbitrary state in terms of the measurement probabilities.

For $D = 2$ (qubits) we can write

$$\Pi_\alpha = \frac{1}{2}(I + \vec{n}_\alpha \cdot \vec{\sigma}) \, , \tag{19}$$

and the universal value, $\mu = 1/\sqrt{3}$, of the inner product translates to

$$\vec{n}_\alpha \cdot \vec{n}_\beta = -\frac{1}{3} \quad \text{for } \alpha \neq \beta. \tag{20}$$

This means that the four Bloch vectors lie at the vertices of a tetrahedron. One choice is

$$
\begin{aligned}
&\vec{n}_1 = \vec{e}_3 \;, \qquad \theta = 0,\ \phi \text{ arbitrary},\\
&\vec{n}_2 = \sqrt{\frac{8}{9}}\vec{e}_1 - \frac{1}{3}\vec{e}_3 \;, \qquad \cos\theta = -1/3,\ \cos(\theta/2) = 1/\sqrt{3},\ \sin(\theta/2) = \sqrt{2/3},\ \phi = 0,\\
&\vec{n}_3 = -\sqrt{\frac{2}{9}}\vec{e}_1 + \sqrt{\frac{2}{3}}\vec{e}_2 - \frac{1}{3}\vec{e}_3 \;,\\
&\qquad\quad \cos\theta = -1/3,\ \cos(\theta/2) = 1/\sqrt{3},\ \sin(\theta/2) = \sqrt{2/3},\ \phi = 2\pi/3,\\
&\vec{n}_4 = -\sqrt{\frac{2}{9}}\vec{e}_1 - \sqrt{\frac{2}{3}}\vec{e}_2 - \frac{1}{3}\vec{e}_3 \;,\\
&\qquad\quad \cos\theta = -1/3,\ \cos(\theta/2) = 1/\sqrt{3},\ \sin(\theta/2) = \sqrt{2/3},\ \phi = -2\pi/3.
\end{aligned}
\tag{21}
$$

Using the convention

$$
|\vec{n}\rangle = e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle \;, \tag{22}
$$

we have

$$
\begin{aligned}
|\vec{n}_1\rangle &= |0\rangle \;,\\
|\vec{n}_2\rangle &= \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \;,\\
|\vec{n}_3\rangle &= e^{-i\pi/3}\frac{1}{\sqrt{3}}|0\rangle + e^{i\pi/3}\sqrt{\frac{2}{3}}|1\rangle \;,\\
|\vec{n}_4\rangle &= e^{i\pi/3}\frac{1}{\sqrt{3}}|0\rangle + e^{-i\pi/3}\sqrt{\frac{2}{3}}|1\rangle \;.
\end{aligned}
\tag{23}
$$

For $D = 3$ (qutrits), we can use the following nine pure states to construct the one-dimensional projectors (I believe Bill Wootters thought up these states; I certainly got the idea from him):

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{2}}\big(|1\rangle + |2\rangle\big)\ , \qquad \theta=\pi/2,\ \phi=\pi/4,\ \chi_1=0,\ \chi_2=0,\\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}\big(\omega|1\rangle + \omega^*|2\rangle\big)\ , \qquad \theta=\pi/2,\ \phi=\pi/4,\ \chi_1=2\pi/3,\ \chi_2=4\pi/3,\\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}\big(\omega^*|1\rangle + \omega|2\rangle\big)\ , \qquad \theta=\pi/2,\ \phi=\pi/4,\ \chi_1=4\pi/3,\ \chi_2=2\pi/3,\\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}\big(|1\rangle + |3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=0,\ \chi_1=0,\ \chi_2 \text{ arbitrary},\\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}\big(\omega|1\rangle + \omega^*|3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=0,\ \chi_1=4\pi/3,\ \chi_2 \text{ arbitrary},\\
|\psi_6\rangle &= \frac{1}{\sqrt{2}}\big(\omega^*|1\rangle + \omega|3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=0,\ \chi_1=2\pi/3,\ \chi_2 \text{ arbitrary},\\
|\psi_7\rangle &= \frac{1}{\sqrt{2}}\big(|2\rangle + |3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=\pi/2,\ \chi_1 \text{ arbitrary},\ \chi_2=0,\\
|\psi_8\rangle &= \frac{1}{\sqrt{2}}\big(\omega|2\rangle + \omega^*|3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=\pi/2,\ \chi_1 \text{ arbitrary},\ \chi_2=4\pi/3,\\
|\psi_9\rangle &= \frac{1}{\sqrt{2}}\big(\omega^*|2\rangle + \omega|3\rangle\big)\ , \qquad \theta=\pi/4,\ \phi=\pi/2,\ \chi_1 \text{ arbitrary},\ \chi_2=2\pi/3,
\end{aligned}
\tag{24}
$$

where

$$\omega \equiv e^{2\pi i/3} \tag{25}$$

and the coördinates are those of the Bloch-sphere-like representation of qutrit states. The corresponding Bloch vectors are

$$
\begin{aligned}
\vec{n}_1 &= \frac{\sqrt{3}}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_8 \ , \\
\vec{n}_2 &= -\frac{\sqrt{3}}{4}\vec{e}_1 + \frac{3}{4}\vec{e}_2 + \frac{1}{2}\vec{e}_8 \ , \\
\vec{n}_3 &= -\frac{\sqrt{3}}{4}\vec{e}_1 - \frac{3}{4}\vec{e}_2 + \frac{1}{2}\vec{e}_8 \ , \\
\vec{n}_4 &= \frac{\sqrt{3}}{4}\vec{e}_3 + \frac{\sqrt{3}}{2}\vec{e}_4 - \frac{1}{4}\vec{e}_8 \ , \\
\vec{n}_5 &= \frac{\sqrt{3}}{4}\vec{e}_3 - \frac{\sqrt{3}}{4}\vec{e}_4 + \frac{3}{4}\vec{e}_5 - \frac{1}{4}\vec{e}_8 \ , \\
\vec{n}_6 &= \frac{\sqrt{3}}{4}\vec{e}_3 - \frac{\sqrt{3}}{4}\vec{e}_4 - \frac{3}{4}\vec{e}_5 - \frac{1}{4}\vec{e}_8 \ , \\
\vec{n}_7 &= -\frac{\sqrt{3}}{4}\vec{e}_3 + \frac{\sqrt{3}}{2}\vec{e}_6 - \frac{1}{4}\vec{e}_8 \ , \\
\vec{n}_8 &= -\frac{\sqrt{3}}{4}\vec{e}_3 - \frac{\sqrt{3}}{4}\vec{e}_6 + \frac{3}{4}\vec{e}_7 - \frac{1}{4}\vec{e}_8 \ , \\
\vec{n}_9 &= -\frac{\sqrt{3}}{4}\vec{e}_3 - \frac{\sqrt{3}}{4}\vec{e}_6 - \frac{3}{4}\vec{e}_7 - \frac{1}{4}\vec{e}_8 \ .
\end{aligned}
\tag{26}
$$

By virtue of

$$\frac{1}{4} = \mu^2 = \mathrm{tr}(\Pi_\alpha \Pi_\beta) = \frac{1}{3}(1 + 2\vec{n}_\alpha \cdot \vec{n}_\beta) \qquad \text{for } \alpha \neq \beta, \tag{27}$$

we have that that these vectors satisfy

$$\vec{n}_\alpha \cdot \vec{n}_\beta = -\frac{1}{8} \quad \text{for } \alpha \neq \beta. \tag{28}$$

Notice that you can't do the $D = 4$ case by taking the tensor product of the tetrahedral vectors for two qubits.