To: *To Whom It May Concern*
From: *C. M. Caves*
Subject: **The Monty Hall Problem**
1998 September 27; updated 2008 April 17

On the television show *Let's Make a Deal*, the host, Monty Hall, shows the contestant three doors. One door hides the grand prize, while the other two hide nothing. The contestant chooses one of the doors. Monty then opens one of the remaining doors, revealing that it hides nothing. The contestant is then offered the opportunity to stick with his original choice or to switch to the one remaining door. Should he switch?

The probability that the contestant's initial guess is right is

$$p(R) = 1/3 ,$$

and the probability that he is wrong is

$$p(W) = 2/3 .$$

If the contestant's initial guess is right, then after Monty's revelation, the probability that the prize lies behind the one remaining door is

$$p(r|R) = 0 ,$$

and the probability that the prize lies behind the door of the contestant's original guess is

$$p(g|R) = 1 .$$

The corresponding probabilities in the case that the contestant's initial guess is wrong are

$$p(r|W) = 1 , \qquad p(g|W) = 0 .$$

Thus the probabilities for the prize to lie behind the remaining and guess doors are

$$\begin{aligned} p(r) &= p(r|R)p(R) + p(r|W)p(W) = p(W) = 2/3 , \\ p(g) &= p(g|R)p(R) + p(g|W)p(W) = p(R) = 1/3 . \end{aligned}$$

The contestant should always change his guess. This analysis can be summarized by saying that the contestant picks the right door 1/3 of the time, in which case he ought to stick with his initial guess, but he picks the wrong door 2/3 of the time, in which case he ought to change his guess.

This scenario can easily be generalized to the situation where there are N doors, the contestant guesses one, Monty opens n doors that don't hide the prize, and the contestant is given the chance to switch to any of the $N - n - 1$ remaining doors. Here we have

$$p(R) = \frac{1}{N} , \qquad p(W) = 1 - \frac{1}{N} .$$

If the contestant's initial guess is right, the probability that the prize lies behind a particular remaining door is

$$p(r|R) = 0 \; ,$$

and the probability that it lies behind the original-guess door is

$$p(g|R) = 1 \; .$$

The corresponding probabilities in the case that the contestant's original guess is wrong are

$$p(r|W) = \frac{1}{N-n-1} \; , \qquad p(g|W) = 0 \; .$$

The resulting unconditioned probabilities are

$$\begin{aligned} p(r) &= p(r|R)p(R) + p(r|W)p(W) = \frac{1}{N-n-1}p(W) = \frac{N-1}{N(N-n-1)} \; , \\ p(g) &= p(g|R)p(R) + p(g|W)p(W) = p(R) = \frac{1}{N} \; . \end{aligned}$$

As long as $n \neq 0$, then the contestant should switch to one of the remaining doors. The opening of the other n doors always supplies cogent information, which makes it wise to switch.

You could get at this result in the following way, which doesn't bother with going through the conditional probabilities for whether the initial guess is right or wrong. I think it's clearer to go through the conditional probabilities, but others might not. The probability for the initial guess being right is unaffected by Monty's opening other doors, so

$$p(g) = p(R) = \frac{1}{N} \; .$$

The remaining closed doors after n are opened are clearly symmetric, so they all have the same probability $p(r)$ for hiding the prize. There being $N-n-1$ of these remaining doors, we have by normalization,

$$p(g) + (N-n-1)p(r) = 1 \qquad \Longrightarrow \qquad p(r) = \frac{1-p(g)}{N-n-1} = \frac{N-1}{N(N-n-1)} \; .$$

In this way of doing things, the enhanced probability of the remaining closed doors is a simple consequence of reducing the number of possibilities from $N-1$ to $N-n-1$ when Monty opens n doors.

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I'm adding a bit here, nearly a decade after the original write-up, because of attention drawn to the Monty Hall problem by a *New York Times* article by John Tierney, which ran on 2008 April 8.

The thing that makes the Monty Hall problem interesting is that for many—perhaps most—people, intuition gets it wrong. Most people conclude that the probabilities for the two remaining doors are equal, so there's no incentive to switch from one's initial choice. Of course, you can see this conclusion is wrong by doing the math, although most people couldn't, or by coming up with the simple argument at the end of the second paragraph. Either way, I suspect that one's intuition hasn't been improved much.

The point of introducing more doors is to put the problem in a context where intuition can easily see that switching is the best option. The hope is that this improved intuition will filter down to the $N = 3$ case. Suppose, for example, that there there are $N = 100$ doors, and after your initial choice, Monty opens $n = 98$ other doors to reveal nothing behind them. Then it is obvious that he has almost told you that the prize is behind the one door he didn't open. The only time that he hasn't told you where the prize is hidden is in the 1% of cases where you picked the prize door initially. There is an overwhelming incentive to switch, because 99% of the time, the other remaining door conceals the prize. Once you have internalized this extreme case, it's much easier to understand why you should switch when there are only three doors.

This is an example of a kind of reasoning that is extremely valuable: test your thinking in an extreme case where the answer is obvious; if your thinking is wrong when the answer is obvious, you had better fix it.