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Subject: **Doubly stochastic operations on qubits**
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Consider a *trace-preserving* quantum operation $\mathcal{A}$, i.e.,

$$\mathrm{tr}\big(\mathcal{A}(\rho)\big) = 1 \quad \text{for all } \rho \quad \Longleftrightarrow \quad \mathcal{A}^{\times}(1) = 1 \ ,$$

which is also *unital*, i.e.,

$$\mathcal{A}(1) = 1 \ .$$

A trace-preserving, unital operation is called *doubly stochastic*. Any convex sum of unitary operations, i.e., an operation of the form

$$\mathcal{A} = \sum_\alpha \lambda_\alpha U_\alpha \otimes U_\alpha^\dagger \ ,$$

with $\lambda_\alpha \geq 0$ and

$$\sum_\alpha \lambda_\alpha = 1 \ ,$$

is doubly stochastic. Our objective is to show that for operations on qubits, any doubly stochastic quantum operation is a convex sum of unitaries. Thus we now restrict $\mathcal{A}$ to be an operation on a single qubit.

Consider a density operator

$$\rho = \frac{1}{2}(1 + \vec{S}\cdot\vec{\sigma}) = \frac{1}{2}(1 + S_j\sigma_j) \ ,$$

where $\vec{S} = \vec{e}_j S_j$ and $\vec{\sigma} = \vec{e}_j\sigma_j$. The operation maps this density operator to

$$\mathcal{A}(\rho) = \frac{1}{2}\big(1 + S_j\mathcal{A}(\sigma_j)\big) \ .$$

The linearity of $\mathcal{A}$ then implies that

$$\mathcal{A}(\sigma_j) = A_{jk}\sigma_k$$

for some real 3×3 matrix A. Thus we have

$$\mathcal{A}(\rho) = \frac{1}{2}\big(1 + S_j A_{jk}\sigma_k\big) = \frac{1}{2}\big(1 + \vec{S}\cdot A\vec{\sigma}\big) = \frac{1}{2}\big(1 + A^T\vec{S}\cdot\vec{\sigma}\big) \ .$$

We assume here that A has positive determinant on the grounds that $\mathcal{A}$ must arise continuously from the identity operation.

The polar decomposition of A is $A = GR$, where R is a three-dimensional rotation (unit-determinant orthogonal) matrix and $G = \sqrt{AA^T}$ is a positive, real symmetric matrix. (This form of the polar decomposition assumes that A has a positive determinant; if we allowed A to have a negative determinant, then R would have to be replaced by PR, P being a reflection.) Notice first that

$$R\vec{\sigma} = U_R^\dagger \vec{\sigma} U_R \quad \Longleftrightarrow \quad U_R^\dagger \sigma_j U_R = R_{jk}\sigma_k \ ,$$

where U_R is the unitary operator that corresponds to the rotation R. Now let $\vec{n}_j = \vec{e}_k Q_{kj}$ be the orthogonal (right-handed) eigenvectors of G, Q being a rotation matrix, and let α_j be the corresponding (nonnegative) eigenvalues. Thus we have (where we must now drop the summation convention for expressions involving the eigenvalues)

$$\sum_{lk} \vec{e}_l G_{lk} Q_{kj} = G\vec{n}_j = \alpha_j \vec{n}_j = \alpha_j \sum_l \vec{e}_l Q_{lj} \quad \Longleftrightarrow \quad \sum_k G_{lk} Q_{kj} = \alpha_j Q_{lj} \ .$$

Now notice that

$$A\vec{\sigma} = GR\vec{\sigma} = U_R^\dagger (G\vec{\sigma}) U_R = \sum_j U_R^\dagger (\vec{\sigma}\cdot\vec{n}_j) G\vec{n}_j U_R = U_R^\dagger \Bigg(\sum_j \alpha_j \vec{n}_j (\vec{\sigma}\cdot\vec{n}_j) \Bigg) U_R \ . \tag{1}$$

[If we allowed A to have a negative determinant, G would be replaced by GP in Eq. (1). Choosing P to be defined by $P\vec{n}_j = \epsilon_j \vec{n}_j$, where $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$, we would find the only consequence to be a change in the sign of α_3 in what follows.] This allows us to write the action of the quantum operation as

$$\begin{aligned} \mathcal{A}(\rho) &= \frac{1}{2}\big(1 + \vec{S}\cdot A\vec{\sigma}\big) \\ &= U_R^\dagger \Bigg(\frac{1}{2}\Bigg(1 + \sum_j \alpha_j (\vec{S}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_j) \Bigg) \Bigg) U_R \ . \end{aligned}$$

When $\vec{S} = \vec{n}_j$, we get

$$\mathcal{A}(\rho) = U_R^\dagger \bigg(\frac{1}{2}\big(1 + \alpha_j \vec{\sigma}\cdot\vec{n}_j\big) \bigg) U_R \ ,$$

from which we can see that the eigenvalues must satisfy $0 \le \alpha_j \le 1$. Indeed, the requirement that $\mathcal{A}$ map positive operators to positive operators is equivalent to $0 \le G \le 1$.

The operation $\mathcal{A}$ maps any spherical surface within the Bloch sphere to an ellipsoidal surface that lies inside the spherical surface, followed by the rotation R^{-1}. The principal axes of the ellipsoidal surface are the $\vec{n}_j$, and the principal radii are the eigenvalues α_j. Let us define the operation that describes the contraction onto ellipsoidal surfaces:

$$\mathcal{B}(\rho) \equiv \frac{1}{2}\Bigg(1 + \sum_j \alpha_j (\vec{S}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_j) \Bigg) \ .$$

The action of $\mathcal{A}$ is obtained by composing $\mathcal{B}$ with the unitary transformation $U_R^\dagger$.

Now we're set. Notice that

$$(\vec{\sigma}\cdot\vec{n}_k)(\vec{\sigma}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_k) = \begin{cases} \vec{\sigma}\cdot\vec{n}_j\ , & \text{for } j=k, \\ -\vec{\sigma}\cdot\vec{n}_j\ , & \text{for } j\neq k, \end{cases}$$

which is just the statement that a 180° rotation about $\vec{n}_k$ inverts the two axes orthogonal to $\vec{n}_k$ while leaving $\vec{n}_k$ fixed. Defining $\vec{\sigma}\cdot\vec{n}_0 \equiv 1$, we can use this result to write

$$\begin{aligned}\sum_{\alpha,j}\lambda_\alpha(\vec{S}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_\alpha)(\vec{\sigma}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_\alpha) &= \underbrace{(\lambda_0+\lambda_1-\lambda_2-\lambda_3)}_{\equiv\,\alpha_1}(\vec{S}\cdot\vec{n}_1)(\vec{\sigma}\cdot\vec{n}_1) \\ &\quad + \underbrace{(\lambda_0-\lambda_1+\lambda_2-\lambda_3)}_{\equiv\,\alpha_2}(\vec{S}\cdot\vec{n}_2)(\vec{\sigma}\cdot\vec{n}_2) \\ &\quad + \underbrace{(\lambda_0-\lambda_1-\lambda_2+\lambda_3)}_{\equiv\,\alpha_3}(\vec{S}\cdot\vec{n}_3)(\vec{\sigma}\cdot\vec{n}_3) \\ &= \sum_j \alpha_j(\vec{S}\cdot\vec{n}_j)(\vec{\sigma}\cdot\vec{n}_j)\ .\end{aligned} \tag{2}$$

We can now write the action of $\mathcal{B}$ in the form

$$\mathcal{B}(\rho) = \sum_\alpha \lambda_\alpha(\vec{\sigma}\cdot\vec{n}_\alpha)\rho(\vec{\sigma}\cdot\vec{n}_\alpha)\ , \tag{3}$$

provided that the coefficients satisfy

$$\sum_\alpha \lambda_\alpha = 1\ . \tag{4}$$

Equation (4), together with the definitions in Eq. (2), allows us to invert the relation between the α's and the λ's to get

$$\begin{aligned}\lambda_0 &= \frac{1}{4}\big(1+\alpha_1+\alpha_2+\alpha_3\big)\ , \\ \lambda_1 &= \frac{1}{4}\big(1+\alpha_1-\alpha_2-\alpha_3\big)\ , \\ \lambda_2 &= \frac{1}{4}\big(1-\alpha_1+\alpha_2-\alpha_3\big)\ , \\ \lambda_3 &= \frac{1}{4}\big(1-\alpha_1-\alpha_2+\alpha_3\big)\ .\end{aligned}$$

Written in the abstract, Eq. (3) says that $\mathcal{B}$ has the form

$$\mathcal{B} = \sum_\alpha \lambda_\alpha(\vec{\sigma}\cdot\vec{n}_\alpha)\otimes(\vec{\sigma}\cdot\vec{n}_\alpha)\ . \tag{4}$$

The operators $\vec{\sigma}\cdot\vec{n}_\alpha$ are orthogonal (unitary) operators. Thus Eq. (4) is the eigendecomposition of $\mathcal{B}$ relative to the left-right action. The operators $\vec{\sigma}\cdot\vec{n}_\alpha/\sqrt{2}$ are the normalized eigenoperators of $\mathcal{B}$, with corresponding eigenvalues $2\lambda_\alpha$. The condition in Eq. (4) is true for any trace-preserving operation, i.e.,

$$2\sum_\alpha \lambda_\alpha = \mathrm{Tr}(\mathcal{B}) = \mathrm{Tr}(\mathcal{B}^\times) = \mathrm{tr}\big(\mathcal{B}^\times(1)\big) = \mathrm{tr}(1) = 2\ .$$

We haven't yet used the requirement that $\mathcal{A}$ (and $\mathcal{B}$) be *completely positive*. Complete positivivity is equivalent to the requirement that $\mathcal{B}$ be positive relative to the left-right action or, equivalently, that its eigenvalues be nonnegative, i.e., that $\lambda_j \geq 0$. These further requirements on the eigenvalues α_j of G are examples of how complete positivity is a stronger requirement than that the operation map positive operators to positive operators.

This brings us to the finish, for we have shown that we can write $\mathcal{A}$ as a convex combination of unitary operations:

$$\mathcal{A} = \sum_\alpha \lambda_\alpha U_R^\dagger(\vec{\sigma}\cdot\vec{n}_\alpha)\otimes(\vec{\sigma}\cdot\vec{n}_\alpha)U_R\ . \tag{5}$$

Indeed, we have the stronger result that $\mathcal{A}$ can be written as a convex combination of the identity transformation and 180° rotations about three orthogonal axes, followed by an overall rotation.