How to perform the coherent measurement of a curved phase space by continuous isotropic measurement

Spin and the Kraus-operator geometry of SL(2,C)

Carlton M. Caves

Distinguished Professor Emeritus, University of New Mexico

Chris Jackson

This is work carried out with
Christopher S. Jackson,
formerly a postdoc at UNM,
now a postdoc at Sandia Livermore.

C. S. Jackson and C. M. Caves, "How to perform the coherent measurement of a curved phase space by continuous isotropic measurement. I. Spin and the Kraus-operator geometry of SL(2,C)," arXiv:2107.12396 [quant-ph].

FOCUS ON THIS.


Chris Jackson

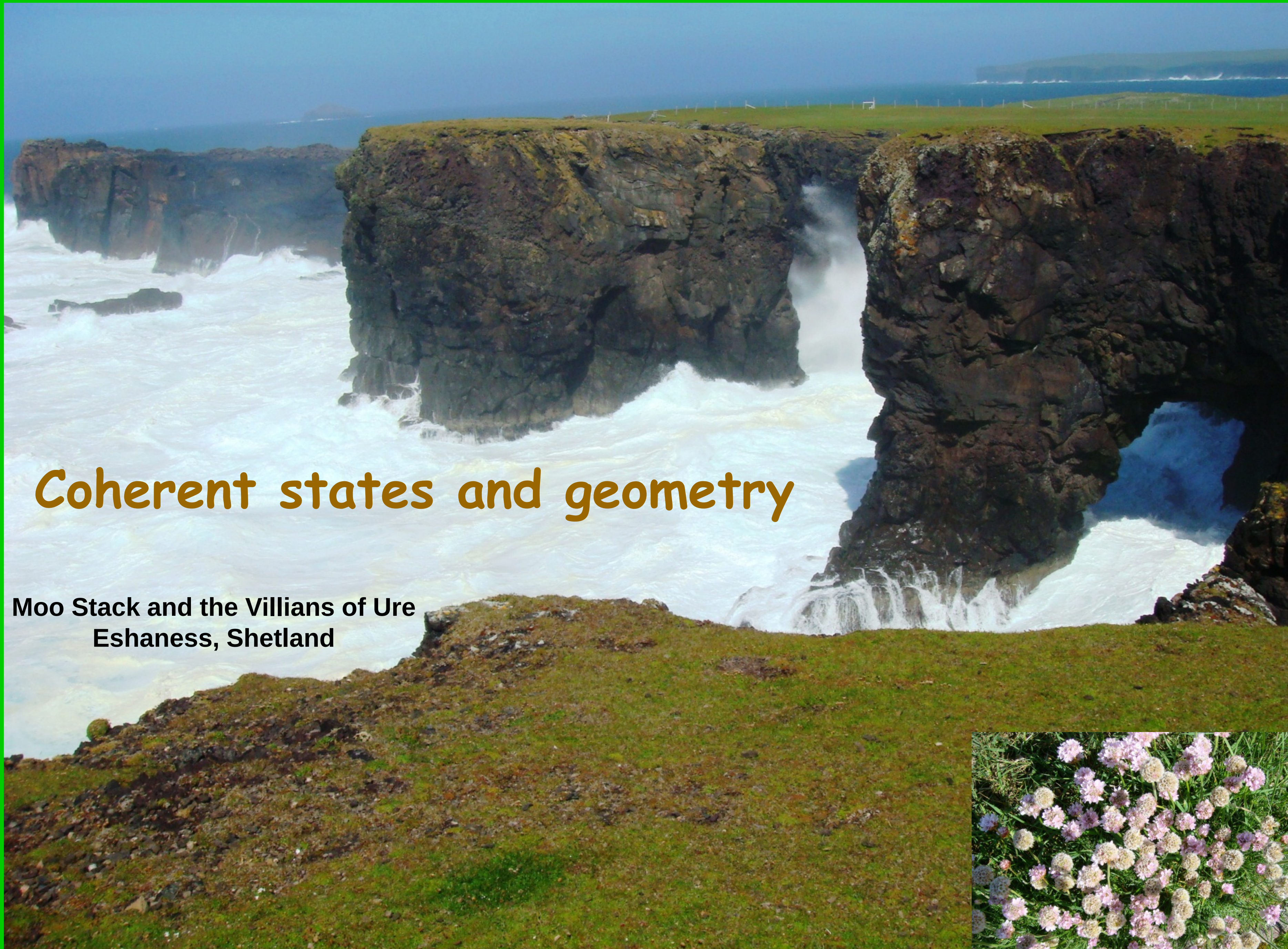
From the acknowledgments: This article is the first in a series describing CSJ's vision for understanding curved phase spaces and their role in physics and quantum information. … CMC is honored to be a co-author, provided it takes nothing away from CSJ's accomplishment.

Destination: What "strong" measurement corresponds to weak, continuous, measurement of (noncommuting) observables?
Journey: Exploring the space of Kraus operators and POVMs that goes with phase space.

C. S. Jackson and C. M. Caves, "How to perform the coherent measurement of a curved phase space by continuous isotropic measurement. II. The Kraus-operator geometries of complex semisimple Lie groups," drafted, but needs retrofitting with present understanding.

Much more.

Coherent states and geometry



Moo Stack and the Villians of Ure
Eshaness, Shetland

Coherent states

Glauber coherent states of a bosonic mode
Lie group: Weyl-Heisenberg

$$[iQ, -iP] = i1$$

Spin coherent states (SCS)
Lie group: SU(2)

$$[-iJ_k, -iJ_l] = -i\epsilon_{klm} J_m$$

CS are ground states of "easy" integrable Hamiltonians

$$\frac{(P-p)^2 + (Q-q)^2}{2}|\alpha\rangle = \frac{1}{2}|\alpha\rangle$$

$$\alpha = (q + ip)/\sqrt{2}$$

$$-(B\hat{\mathbf{n}})\cdot \mathbf{J}\,|j,\hat{\mathbf{n}}\rangle = -jB|j,\hat{\mathbf{n}}\rangle$$

$$\hat{\mathbf{n}} = \hat{\mathbf{z}}\cos\theta + (\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi)\sin\theta$$

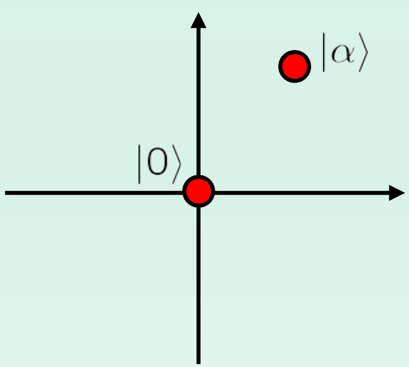
Lie-group displacement

$$D(\alpha) = e^{-iqP+ipQ}$$

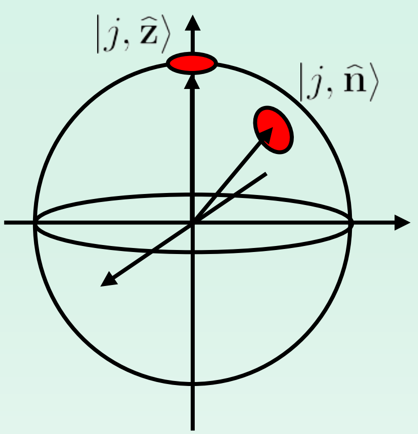
$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\hat{\mathbf{n}}) = e^{-i\theta\left(J_y\cos\phi - J_x\sin\phi\right)}$$

$$|j,\hat{\mathbf{n}}\rangle = D(\hat{\mathbf{n}})|j,\hat{\mathbf{z}}\rangle$$



Phase space



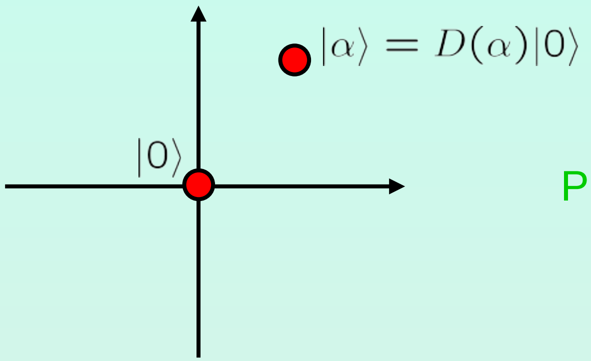
Coherent states

Glauber coherent states of a bosonic mode
Lie group: Weyl-Heisenberg

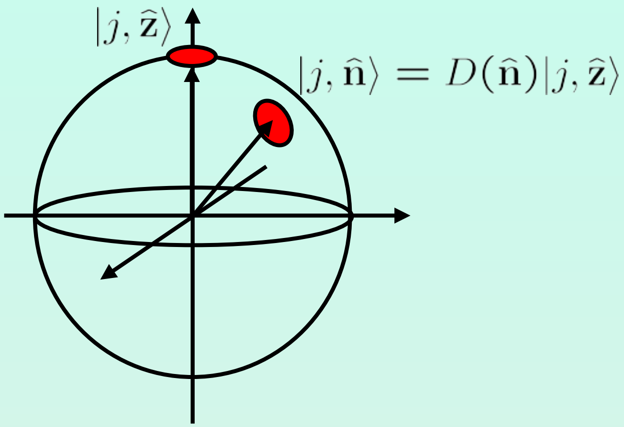
$$[iQ, -iP] = i1$$

Spin coherent states (SCS)
Lie group: SU(2)

$$[-iJ_k, -iJ_l] = -i\epsilon_{klm} J_m$$



Phase space



Ladder operators (Cartan-Weyl basis)

$$D(\alpha) a D(\alpha)^\dagger |\alpha\rangle = (a - \alpha)|\alpha\rangle = 0$$

$$a = (Q + iP)/\sqrt{2}$$

$$D(\hat{\mathbf{n}}) J_+ D(\hat{\mathbf{n}})^\dagger |j, \hat{\mathbf{n}}\rangle = 0$$

$$J_+ = J_x + iJ_y$$

Tensor miracle (classical limit)

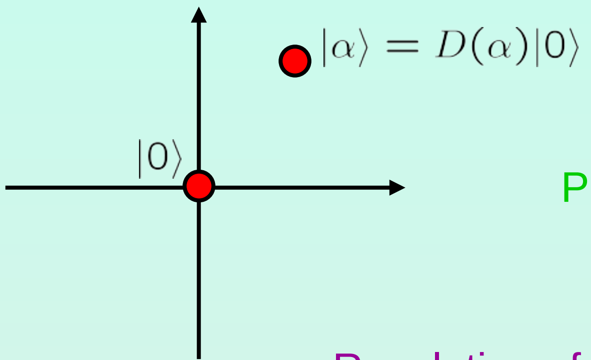
$$|\alpha\rangle^{\otimes N} \otimes |\sqrt{s}\alpha\rangle \cong |\alpha\sqrt{N+s}\rangle$$

$$|\tfrac{1}{2}, \hat{\mathbf{n}}\rangle^{\otimes 2j} \cong |j, \hat{\mathbf{n}}\rangle$$

Coherent states

Glauber coherent states of a bosonic mode
Lie group: Weyl-Heisenberg

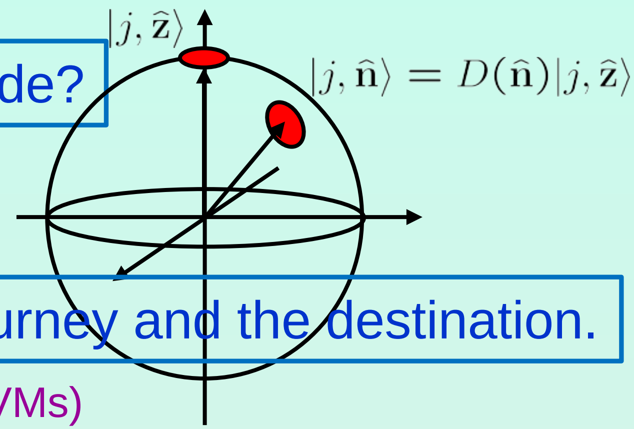
$$[iQ, -iP] = i1$$



Spin coherent states (SCS)
Lie group: SU(2)

$$[-iJ_k, -iJ_l] = -i\epsilon_{klm}J_m$$

What's inside?



Phase space

The journey and the destination.

Resolution of the identity (CS POVMs)

$$\frac{1}{\pi}\int_C d^2\alpha\,|\alpha\rangle\langle\alpha| = 1$$

$$(2j+1)\int_{S^2}\frac{d\theta\,\sin\theta\,d\phi}{4\pi}|j,\hat{\mathbf{n}}\rangle\langle j,\hat{\mathbf{n}}| = 1_j$$

CS POVM: $\left\{\frac{1}{\pi}d^2\alpha\,|\alpha\rangle\langle\alpha|\right\}$

SCS POVM: $\left\{(2j+1)\frac{d\theta\,\sin\theta\,d\phi}{4\pi}|j,\hat{\mathbf{n}}\rangle\langle j,\hat{\mathbf{n}}|\right\}$

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

$$Q^{(j)}(\hat{\mathbf{n}}) = (2j+1)\langle j,\hat{\mathbf{n}}|\rho|j,\hat{\mathbf{n}}\rangle$$

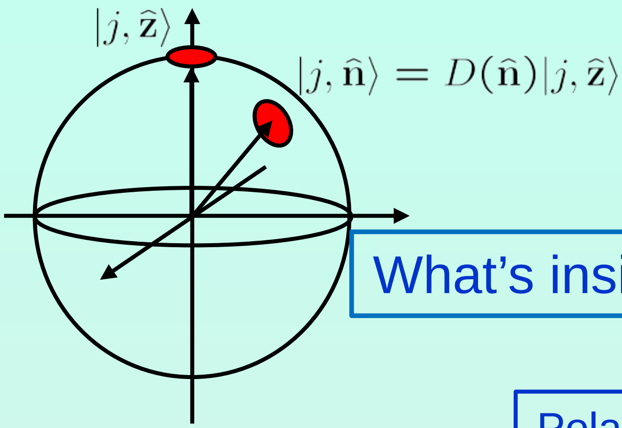
Measure by (single-shot) heterodyne: simultaneous, isotropic measurement of *Q* and *P*.

Measure by ?

Continuous (weak) isotropic measurement of the three spin components.

E. Shojaee, C. S. Jackson, C. A. Riofrío, A. Kalev, and I. H. Deutsch, PRL **121**, 130404 (2018).

Geometry



What's inside?

What's inside? A type-IV symmetric space, in this case a 3-hyperboloid, consisting of concentric 2-spheres of thermal states with inverse temperature 2*a*, running from the identity operator at *a* = 0 to the 2-sphere of SCSs at *a* = infinity.

Polar decomposition

Cartan (singular-value) decomposition

Overall Kraus operator: $K = W\sqrt{E} = WU^\dagger e^{aJ_z}U = Ve^{aJ_z}U$

Postmeasurement unitary

Premeasurement unitary

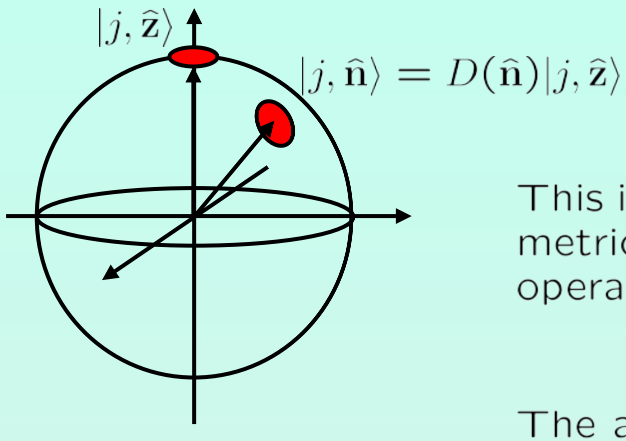
POVM element: $E = K^\dagger K = U^\dagger e^{2aJ_z}U = D(\hat{\mathbf{n}})e^{2aJ_z}D(\hat{\mathbf{n}})^\dagger = e^{2a\hat{\mathbf{n}}\cdot\mathbf{J}}$

U can be restricted to displacements.

Thermal state: inverse temperature 2*a*

Our analysis of continuous isotropic measurements is Kraus-operator-centric—state-independent and representation-independent and thus geometric. Kraus operators are the triple entendre (CSJ) or trinity (CMC) of quantum theory, simultaneously representing states, measurements (POVMs), and transformations (processes). To include all Kraus operators, one attaches an SU(2) fiber of postmeasurement unitaries at every point in the 3-hyperboloid. One is then studying the Kraus-operator geometry of the complexification of SU(2), which is SL(2,C).

Geometry



What's inside? A type-IV symmetric space, in this case a 3-hyperboloid, consisting of concentric 2-spheres of thermal states with inverse temperature 2*a*, running from the identity operator at *a* = 0 to the 2-sphere of SCSs at *a* = infinity.

This is the *3-hyperboloid of negative constant curvature*. Its metric, coming from the Maurer-Cartan form of the Kraus operators and thus, ultimately, the Killing form of SU(2), is

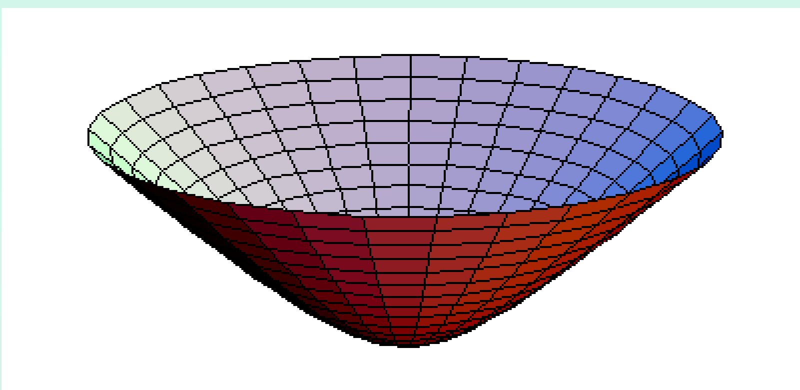
$$ds^2 = da^2 + \sinh^2 a\,(d\theta^2 + \sin^2\theta\, d\phi^2).$$

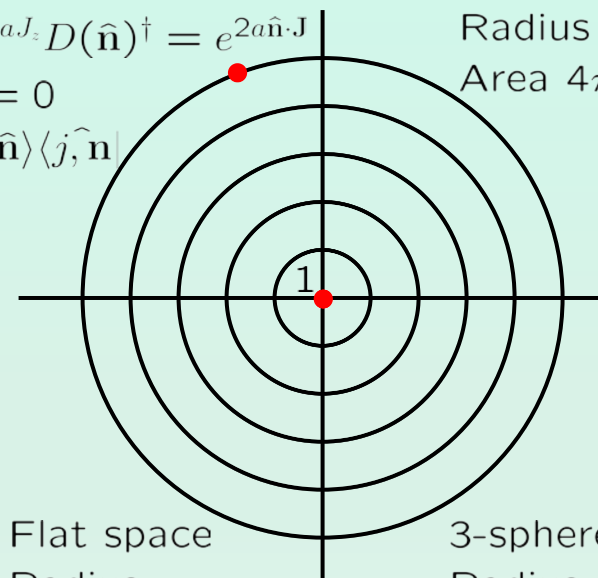
The area of the 2-sphere of radius a is $4\pi\sinh^2 a$.

$E = D(\hat{\mathbf{n}})e^{2aJ_z}D(\hat{\mathbf{n}})^\dagger = e^{2a\hat{\mathbf{n}}\cdot\mathbf{J}}$

$E = 1$ at $a = 0$

$E \underset{a\to\infty}{\longrightarrow} e^{2aj}|j,\hat{\mathbf{n}}\rangle\langle j,\hat{\mathbf{n}}|$





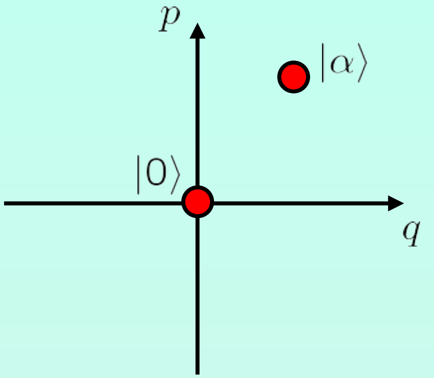
3-hyperboloid
Radius a
Area $4\pi\sinh^2 a$

Flat space
Radius a
Area $4\pi a^2$

3-sphere
Radius a
Area $4\pi\sin^2 a$

To get the geometry right, one must regard the hyperboloid as embedded in Minkowski space, not Euclidean space.

What's misleading about conventional *q*-*p* phase space?



The irreducible tensors used for expanding operators are identical to the displacement operators. This is not true for generalized coherent states and the curved phase spaces of compact semisimple Lie groups.

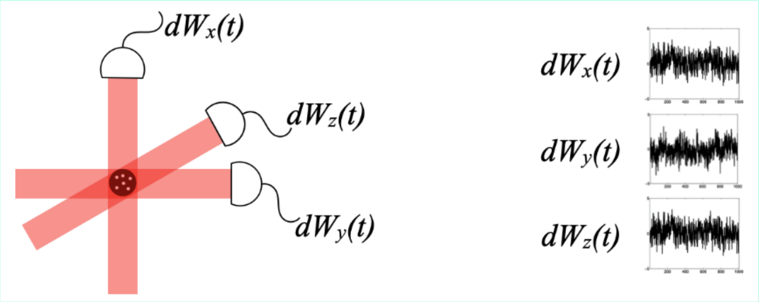
The near commutativity of *q* and *p* gives a nearly trivial operator ordering for constructing phase-space correspondences and quasidistributions. For compact semisimple Lie groups, these projects are best approached in a quite different way.

There is no apparent "inside." Because there is a single-shot heterodyne measurement, there is little motivation to consider weak, continuous measurements. Nonetheless, you should, for the journey it takes through the "inside" symmetric space of displaced thermal states is transformative.

Measuring the SCS POVM

Holstrandir Peninsula overlooking Ísafjarðardjúp
Westfjords, Iceland

Measuring the three spin components weakly and simultaneously



The distinctive feature of our analysis is that it is Kraus-operator-centric.

Gaussian Kraus operator for weak, simultaneous measurement of three spin components lasting time *dt*.

$$\sqrt{d\mu(d\mathbf{W})}\,e^{-\mathbf{J}^2\gamma\,dt}e^{\mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}}$$

$$\gamma = (\text{measurement rate})$$

$$d\mathbf{W} = (\text{vector of measurement outcomes}) = (\text{vector Wiener increment})$$

$$\text{Itô rule: } dW^\mu dW^\nu = \delta^{\mu\nu}dt$$

$$(\text{Gaussian measure}) = d\mu(d\mathbf{W}) \equiv \frac{d(dW^x)d(dW^y)d(dW^z)}{(2\pi dt)^{3/2}}\exp\left(-\frac{d\mathbf{W}\cdot d\mathbf{W}}{2dt}\right)$$

$$\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2 = j(j+1)1_j = (\text{Casimir invariant})$$

$$e^{\mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}} = K(d\mathbf{W}) = \begin{pmatrix}\text{“Kraus operator”}\\ \text{for increment } dt\end{pmatrix}$$

Element of SL(2,C)
Submanifold closure

Weiner-like path integral

QOVM after time *T* is a Wiener-like outcome-path integral.

$$\mathcal{D}\mathcal{Z}[d\mathbf{W}_{[0,T)}] = \mathcal{D}\mu[d\mathbf{W}_{[0,T)}]\, e^{-\gamma T\left(\mathbf{J}^2\odot 1+1\odot\mathbf{J}^2\right)} \circ K[d\mathbf{W}_{[0,T)}] \odot K[d\mathbf{W}_{[0,T)}]^\dagger$$

Overall Kraus operator $K[d\mathbf{W}_{[0,T)}] = K(T) = K(d\mathbf{W}_{T-dt})\cdots K(d\mathbf{W}_{0dt})$

$$K(d\mathbf{W}_t) = e^{\mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}_t}$$

Semisimple unraveling of trace-preserving, unconditioned QOVM

Haar measure

$$\mathcal{Z}_T = \int \mathcal{D}\mathcal{Z}[d\mathbf{W}_{[0,T)}] = \left(e^{-\gamma T\mathbf{J}^2}\odot e^{-\gamma T\mathbf{J}^2}\right)\circ\int_{\mathrm{SL}(2,\mathrm{C})} d\mu(K)\, D_T(K)K\odot K^\dagger$$

It is critical to appreciate that we are not dealing with a probability distribution for outcomes—that would require an initial state—but rather with a distribution over an ensemble of Kraus operators K, drawn from SL(2,C) and labeled by outcomes.

When you see a path integral, you should be thinking diffusion equation and stochastic differential equation (SDE). Deriving the diffusion equation is hard, but worth it, because it teaches about the geometry. Deriving the SDEs is straightforward, but instead of teaching about, is informed by the geometry. Let's do SDEs first.

Stochastic differential equations

QOVM after time T is a Wiener-like outcome-path integral.

$$\mathcal{D}\mathcal{Z}[d\mathbf{W}_{[0,T)}] = \mathcal{D}\mu[d\mathbf{W}_{[0,T)}]\; e^{-\gamma T\left(\mathbf{J}^2\odot 1+1\odot\mathbf{J}^2\right)} \circ K[d\mathbf{W}_{[0,T)}] \odot K[d\mathbf{W}_{[0,T)}]^\dagger$$

Overall Kraus operator $K[d\mathbf{W}_{[0,T)}] = K(T) = K(d\mathbf{W}_{T-dt})\cdots K(d\mathbf{W}_{0dt})$

$$K(d\mathbf{W}_t) = e^{\mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}_t}$$

SDE for K

$$dK\,K^{-1} = K(d\mathbf{W}_t) - 1 = \mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}_t + \tfrac{1}{2}\mathbf{J}^2\gamma\,dt$$

Maurer-Cartan form

$$dK\,K^{-1} - \tfrac{1}{2}(dK\,K^{-1})^2 = \mathbf{J}\cdot\sqrt{\gamma}\,d\mathbf{W}$$

Modified Maurer-Cartan stochastic differential (MMCSD).
Submanifold closure.

To get something useable, one translates the SDE into SDEs for the pieces of the Cartan decomposition,

$$K = V e^{aJ_z} U\,.$$

Stochastic differential equations

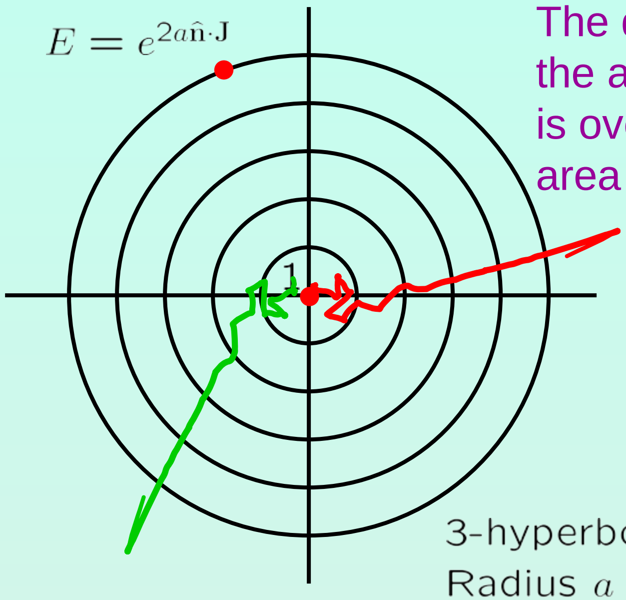
After a few collapse times, 1/γ,
the radial coördinate moves nearly ballistically, with mean and variance proportional to γ*t*,

Radial coördinate

$$da = \gamma\, dt \coth a + \sqrt{\gamma}\, dW_R$$

Ballistic term (Omitted from Shojaee et al.)

Diffusion



The diffusion of *U* in the angular directions is overwhelmed by the area exponentiation.

3-hyperboloid
Radius a
Area $4\pi \sinh^2 a$

the premeasurement unitary freezes out,
thereby picking a direction on the 3-hyperboloid,

Premeasurement unitary

$$dU\,U^{-1} - \tfrac{1}{2}(dU\,U^{-1})^2 = (-iJ_x\sqrt{\gamma}\,dW_T^y + iJ_y\sqrt{\gamma}\,dW_T^x)\operatorname{csch} a$$

Diffusion

and the postmeasurement unitary moves randomly on SU(2).

Postmeasurement unitary

$$dV\,V^{-1} - \tfrac{1}{2}(dV\,V^{-1})^2 = -(-iJ_x\sqrt{\gamma}\,dW_T^y + iJ_y\sqrt{\gamma}\,dW_T^x)\coth a$$

Diffusion

Measuring the SCS POVM

After a few collapse times, $1/\gamma$, the largest eigenvalue of a typical POVM element, $E = U^\dagger e^{2aJ_z} U = e^{2a\mathbf{J}\cdot\mathbf{n}}$, dominates the second-largest eigenvalue by an exponential factor $e^{2a} \sim e^{2\gamma t}$.

The continuous isotropic measurement of spin components collapses to the SCS POVM "almost always" and "in no time at all."

Diffusion equations. Advanced course in geometry

Semisimple unraveling of trace-preserving, unconditioned QOVM

$$\mathcal{Z}_T = \int \mathcal{D}\mathcal{Z}[d\mathbf{W}_{[0,T)}] = \left(e^{-\gamma T \mathbf{J}^2} \odot e^{-\gamma T \mathbf{J}^2}\right) \circ \int_{\mathrm{SL(2,C)}} d\mu(K)\, D_T(K) K \odot K^\dagger$$

Diffusion equation for Kraus-operator distribution $D_t(K)$: $\quad \dfrac{\partial D_t}{\partial t} = \dfrac{\gamma}{2} \Delta[D_t]$

Isotropic measurement Laplacian

SU(2) Killing form

$$\Delta[f] = \delta^{\mu\nu} \widetilde{J}_\mu \Big[\widetilde{J}_\nu[f] \Big] = \widetilde{J}_x \Big[\widetilde{J}_x[f] \Big] + \widetilde{J}_y \Big[\widetilde{J}_y[f] \Big] + \widetilde{J}_z \Big[\widetilde{J}_z[f] \Big]$$

Right-invariant derivatives. Moving basis-vector fields on SL(2,C).

At each *K* diffusion occurs into a 3-submanifold that looks locally like the 3-hyperboloid. This diffusion is nonintegrable and thus explores all of 6-dimensional SL(2,C) because the local 3-submanifolds do not mesh to form a global 3-surface.

Diffusion equations. Advanced course in geometry

Isotropic measurement Laplacian

$$\Delta[f] = \delta^{\mu\nu}\widetilde{J}_\mu\left[\widetilde{J}_\nu[f]\right] = \widetilde{J}_x\left[\widetilde{J}_x[f]\right] + \widetilde{J}_y\left[\widetilde{J}_y[f]\right] + \widetilde{J}_z\left[\widetilde{J}_z[f]\right]$$

To get something useable, one translates the Laplacian into partial derivatives relative to the pieces of the Cartan decomposition,

$$K = V e^{aJ_z} U .$$

Cartan expression for isotropic measurement Laplacian

$$\begin{aligned}
\Delta[f] &= \frac{1}{\sqrt{\det g}}\nabla_\mu\left[\sqrt{\det g}\, g^{\mu\nu}\nabla_\nu[f]\right] \\
&= \frac{1}{\sinh^2 a}\frac{\partial}{\partial a}\left[\sinh^2 a\frac{\partial f(K)}{\partial a}\right] \\
&\quad + \frac{1}{\sinh^2 a}\Big(\nabla_x[\nabla_x[f]]\,(K) + \nabla_y[\nabla_y[f]]\,(K)\Big)
\end{aligned}$$

Beltrami form relative to 3-hyperboloid metric

Fokker-Planck equation for purity distribution $P_t(a)$

$$\frac{\partial}{\partial t}P_t(a) = \frac{\gamma}{2}\frac{\partial}{\partial a}\left[\sinh^2 a\frac{\partial}{\partial a}\left[\frac{1}{\sinh^2 a}P_t(a)\right]\right] = -\gamma\frac{\partial}{\partial a}[\coth a\, P_t(a)] + \frac{\gamma}{2}\frac{\partial^2}{\partial a^2}P_t(a)$$

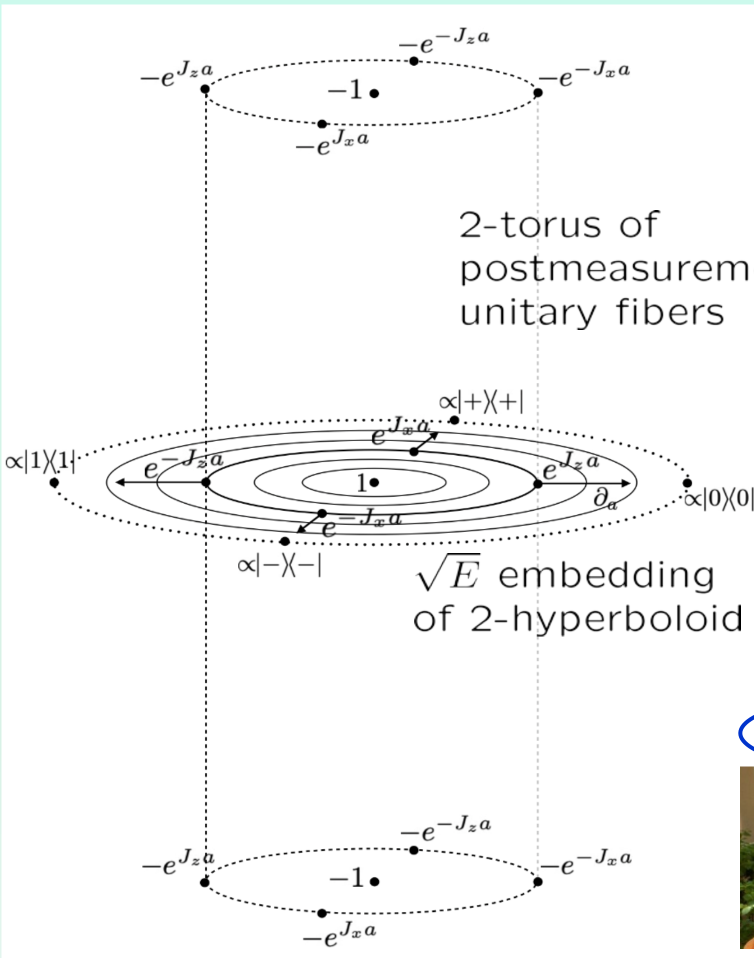
Ballistic term

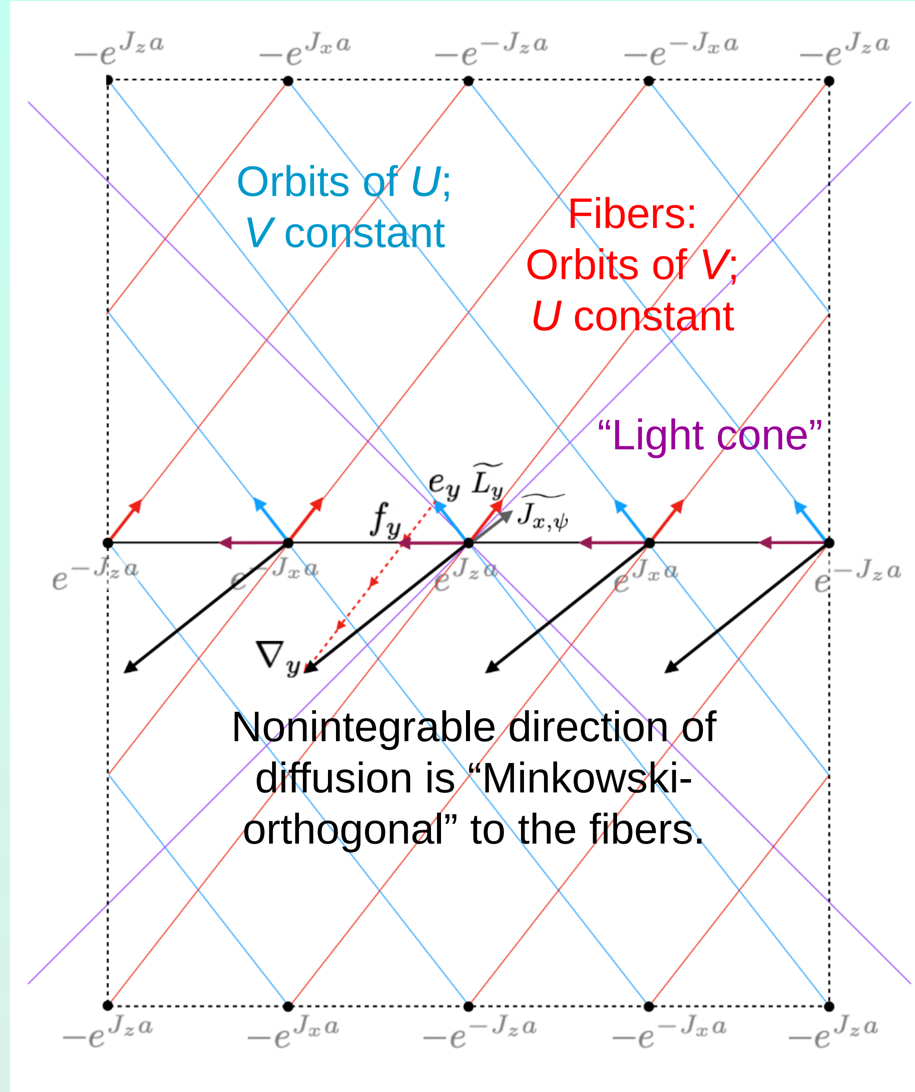
Diffusion

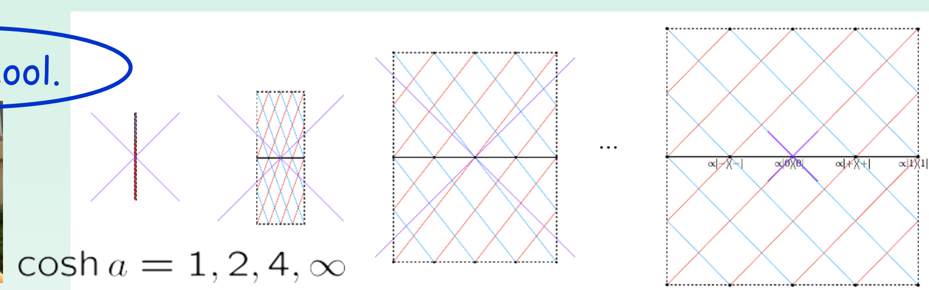
Kraus-operator geometry of SL(2,C)

To draw pictures, suppress two dimensions in the SU(2) postmeasurement-unitary fiber and one dimension in the 3-hyperboloid, effectively working in SL(2,R), with

$$K = Ve^{aJ_z}U = e^{-iJ_y\psi}e^{aJ_z}e^{-iJ_y\phi}.$$







More and much more

Western diamondback rattlesnake
My front yard, Sandia Heights

More

Get out of Hilbert space. The arena of physics, classical and quantum, is the (curved) phase spaces of compact, connected, semi-simple Lie groups.

What about flat, canonical phase space? It's for the bosons, misleadingly special, where phase-space quantum mechanics is already appreciated, but needs to be recast in our language.


Chris Jackson

1. The properties of physical systems are the generators of these Lie groups.
2. Weak, continuous, isotropic measurement of the generators limits to measurement in the (overcomplete) basis of generalized coherent states, thereby identifying the coherent states. Kraus operators drawn from the (complexified) group describe dynamics; Cartan decomposition identifies the radial directions on a type-IV symmetric space, the angular directions in the premeasurement unitaries (displacement operators), and the fiber of postmeasurement unitaries. The phase space of coherent states occupies the boundary of the symmetric space, a space of constant curvature, made up of thermal states with Hamiltonians linear in the generators. Kraus operators are the unifying mathematical object, the quantum trinity, the triple entendre, encompassing states, transformations, and measurements.
3. Putting the Weyl-Heisenberg group and its flat space in the language of curved phase spaces, thus elucidating its universal properties and its special properties.

Much more

Get out of Hilbert space. The arena of physics, classical and quantum, is the (curved) phase spaces of compact, connected, semi-simple Lie groups.

What about flat, canonical phase space? It's for the bosons, misleadingly special, but where phase-space quantum mechanics is already appreciated.

Chris Jackson

4. Harmonic functions span the space of phase-space functions. Associated irreducible tensors span the operator space. Phase-space correspondences and "*s*-ordered" quasiprobabilities, which connect harmonic functions to coherent states, are founded on the "quartic twirl" of generalized coherent states.
5. Quantization is omission of fine-scale harmonic functions (i.e., working in a finite-dimensional irrep). The classical limit is keeping all the harmonic functions, down to the finest scales (i.e., working in an infinite-dimensional irrep). Linear Hilbert-space structure of quantum mechanics arises naturally from the group representation.
6. *Dynamical complexity* (nonlinear dynamics and chaos) is stepping outside the complexified group. Complexity should be measured relative to phase space. Linear evolution, Hamiltonian or dissipative, stays in the group and respects the scales of the harmonic functions. Nonlinear evolution steps outside the group by mixing large and small scales, leading to the sensitivity to initial conditions of classical chaos and to hypersensitivity to perturbation both classically and quantum mechanically.