The right kind of uncertainty principle. Fisher information and Quantum Cramér-Rao bounds

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Quantum Cramér-Rao Bound (QCRB)

Single-parameter estimation: Bound on the mean-square error in estimating a classical parameter that is coupled to a quantum system in terms of the inverse of the quantum Fisher information.

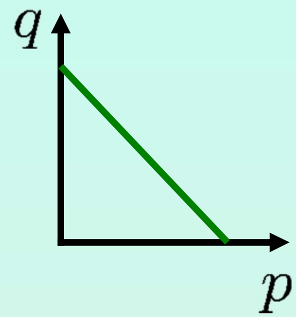
Multi-parameter estimation: Bound on the covariance matrix in estimating a set of classical parameters that are coupled to a quantum system in terms of the inverse of a quantum Fisher-information matrix.

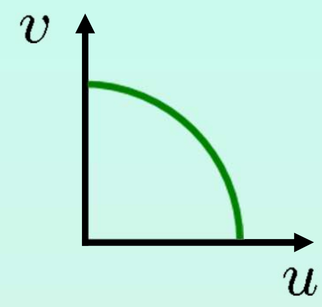
We are only going to consider the single-parameter case. An important distinction is that the single-parameter QCRB is saturable—there is a measurement that achieves the bound—whereas the multi-parameter QCRB is generally not.

Fisher information

Estimating a probability *p* from *N* trials (random walk, polling)

$$
\begin{aligned}
f &= \frac{n}{N} \\
\bar{f} &= \frac{\bar{n}}{N} = p \\
(\Delta f)^2 &= \frac{pq}{N} \qquad q = 1 - p
\end{aligned}
$$





Measuring the "distance" between neighboring probability distributions in units of their distinguishability

$$
\frac{(\delta p)^2}{(\Delta f)^2} = N\frac{(\delta p)^2}{pq} = N\left(\frac{(\delta p)^2}{p} + \frac{(\delta q)^2}{q}\right) = N\big((\delta u)^2 + (\delta v)^2\big)
$$

Fisher information

$$
u = \sqrt{p}\,, \qquad v = \sqrt{q}
$$

Not $|\delta p| + |\delta q|$ or $(\delta p)^2 + (\delta q)^2$ or $(\delta p/p)^2 + (\delta q/q)^2$

(Classical) Cramér-Rao bound

For any parameter ϕ that affects a probability distribution $p_j(\phi)$,

$$\Delta\phi_{\text{est}} \geq \frac{1}{\sqrt{N}}\frac{1}{\sqrt{F}}$$

$$\begin{aligned} F(\phi) &= \left(\begin{array}{c} \text{Fisher} \\ \text{information} \end{array} \right) \\ &= \sum_j \frac{1}{p_j(\phi)} \left(\frac{dp_j(\phi)}{d\phi} \right)^2 \\ &= \sum_j p_j(\phi) \left(\frac{d \ln p_j(\phi)}{d\phi} \right)^2 \end{aligned}$$

Proof of QCRB. Setting

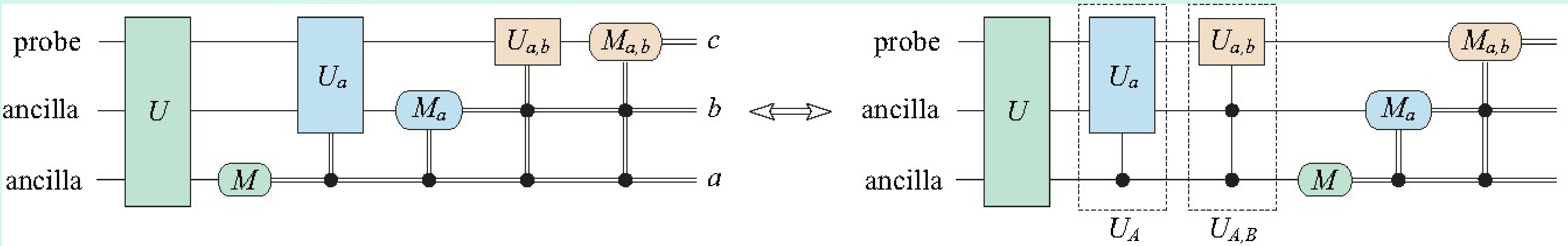
$$H_\gamma(t) = \hbar\gamma h + \tilde{H}(t)$$

Parameter γ

Generator of parameter displacements, h

Everything else, $\tilde{H}$, *including feedback*

For a more traditional proof of the QCRB, including nonunitary parameter displacements, see S. L. Braunstein and C. M. Caves, PRL 72, 3439 (1994).



$$\rho_\gamma(t) = U_\gamma(t)\rho_0 U_\gamma^\dagger(t) \qquad i\hbar\frac{\partial U_\gamma(s)}{\partial s} = H_\gamma(s)U_\gamma(s)$$

$$p(\gamma_{\text{est}}|\gamma) = \text{tr}(E_{\gamma_{\text{est}}}\rho_\gamma(t)) \qquad \int d\gamma_{\text{est}}\, E_{\gamma_{\text{est}}} = I$$

Initial state ρ_0

Intermediate times s, final time t

Evolution operator $U_\gamma(s)$

Estimator POVM $E_{\gamma_{\text{est}}}$

Proof of QCRB. Classical CRB

$$\Delta\gamma_{\text{est}} \equiv \gamma_{\text{est}} - \langle\gamma_{\text{est}}\rangle$$

$$\delta\gamma \equiv \frac{\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} - \gamma = \frac{\Delta\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} + \langle\delta\gamma\rangle$$

Unbiased estimator:

$\langle\gamma_{\text{est}}\rangle = \gamma$ and $d\langle\gamma_{\text{est}}\rangle/d\gamma = 1$ and thus $\langle\delta\gamma\rangle = 0$

$$1 = \int d\gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma) \quad \text{and} \quad \langle\gamma_{\text{est}}\rangle = \int d\gamma_{\text{est}}\, \gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma)$$

Differentiate with respect to γ

$$\begin{aligned}
\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma} &= \int d\gamma_{\text{est}}\, (\gamma_{\text{est}} - \langle\gamma_{\text{est}}\rangle) \frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \\
&= \int d\gamma_{\text{est}}\, \Delta\gamma_{\text{est}} \frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \\
&= \int d\gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma) \Delta\gamma_{\text{est}} \frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \\
&= \left\langle \Delta\gamma_{\text{est}} \frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \right\rangle
\end{aligned}$$

Proof of QCRB. Classical CRB

$$\begin{aligned}\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma} &= \int d\gamma_{\text{est}}\,\Delta\gamma_{\text{est}}\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\\ &= \int d\gamma_{\text{est}}\,p(\gamma_{\text{est}}|\gamma)\Delta\gamma_{\text{est}}\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\\ &= \left\langle\Delta\gamma_{\text{est}}\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\right\rangle\end{aligned}$$

$$\left(\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma}\right)^2 \le \langle(\Delta\gamma_{\text{est}})^2\rangle\left\langle\left(\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\right)^2\right\rangle = \langle(\Delta\gamma_{\text{est}})^2\rangle F(\gamma)$$

Schwarz inequality

Classical Fisher information $F(\gamma)$

$$\delta\gamma \equiv \frac{\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} - \gamma = \frac{\Delta\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} + \langle\delta\gamma\rangle$$

$$\langle(\delta\gamma)^2\rangle = \frac{\langle(\Delta\gamma_{\text{est}})^2\rangle}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|^2} + \langle\delta\gamma\rangle^2 \ge \frac{1}{F(\gamma)} + \langle\delta\gamma\rangle^2 \ge \frac{1}{F(\gamma)}$$

For an unbiased estimator, which has $\langle\gamma_{\text{est}}\rangle = \gamma$, $\langle\delta\gamma\rangle = 0$.

Proof of QCRB. Classical Fisher information

$$\langle(\delta\gamma)^2\rangle = \frac{(\Delta\gamma_{\text{est}})^2}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|^2} + \langle\delta\gamma\rangle^2 \geq \frac{1}{F(\gamma)} + \langle\delta\gamma\rangle^2 \geq \frac{1}{F(\gamma)}$$

$$\begin{aligned} F(\gamma) &\equiv \left\langle \left(\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \right)^2 \right\rangle \\ &= \int d\gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma) \left(\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \right)^2 \\ &= \int d\gamma_{\text{est}} \frac{1}{p(\gamma_{\text{est}}|\gamma)} \left(\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \right)^2 \end{aligned}$$

The Fisher bound $\langle(\delta\gamma)^2\rangle \geq 1/F(\gamma)$ can be attained asympotically in many trials by using maximum-likelihood estimation.

Proof of QCRB. Quantum mechanics

$$H_\gamma(t) = \hbar\gamma h + \tilde{H}(t)$$

$$\rho_\gamma(t) = U_\gamma(t)\rho_0 U_\gamma^\dagger(t)$$

$$i\hbar\frac{\partial U_\gamma(s)}{\partial s} = H_\gamma(s)U_\gamma(s)$$

$$p(\gamma_{\text{est}}|\gamma)) = \text{tr}(E_{\gamma_{\text{est}}}\rho_\gamma(t))$$

$$F(\gamma) = \int d\gamma_{\text{est}}\frac{1}{p(\gamma_{\text{est}}|\gamma)}\left(\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\right)^2$$

$$\begin{aligned}\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} &= \text{tr}\left(E_{\gamma_{\text{est}}}\frac{\partial\rho_\gamma(t)}{\partial\gamma}\right)\\ &= \frac{1}{2}[\text{tr}(E_{\gamma_{\text{est}}}\mathcal{L}_\gamma\rho_\gamma) + \text{tr}(E_{\gamma_{\text{est}}}\mathcal{L}_\gamma\rho_\gamma)^*]\\ &= \text{Re}[\text{tr}(E_{\gamma_{\text{est}}}\mathcal{L}_\gamma\rho_\gamma(t))]\end{aligned}$$

$$\begin{aligned}\text{tr}(E_{\gamma_{\text{est}}}\mathcal{L}_\gamma\rho_\gamma)^* &= \text{tr}(\rho_\gamma\mathcal{L}_\gamma E_{\gamma_{\text{est}}})\\ &= \text{tr}(E_{\gamma_{\text{est}}}\rho_\gamma\mathcal{L}_\gamma)\end{aligned}$$

$$\frac{\partial\rho_\gamma(t)}{\partial\gamma} = \frac{1}{2}[\mathcal{L}_\gamma\rho_\gamma(t) + \rho_\gamma(t)\mathcal{L}_\gamma] = -iU_\gamma(t)[K_\gamma(t),\rho_0]U_\gamma^\dagger(t)$$

Symmetric logarithmic derivative $\mathcal{L}_\gamma$ (Hermitian)

γ-generator K_γ, referred to initial time

$$K_\gamma(t) = iU_\gamma^\dagger(t)\frac{\partial U_\gamma(t)}{\partial\gamma} = \int_0^t ds\, U_\gamma^\dagger(s)hU_\gamma(s) \equiv t\bar{h}$$

If $\tilde{H} = 0$, $U_\gamma(t) = e^{-i\gamma ht}$, $K_\gamma = th$, and $\bar{h} = h$.

Proof of QCRB. Quantum mechanics

The symmetric logarithmic derivative $\mathcal{L}_\gamma$ is natural for the QCRB. For pure state
is directly related to the γ-generator K_γ, which turn out to be optimal.

There is a trick here. One might think that the natural way to define the γ-generator is

$$\mathcal{J}_\gamma(t) = i\frac{\partial U_\gamma(t)}{\partial\gamma}U_\gamma^\dagger(t)\,,$$

but we use instead

$$\mathcal{K}_\gamma(t) = iU_\gamma^\dagger(t)\frac{\partial U_\gamma(t)}{\partial\gamma} = U_\gamma^\dagger(t)\mathcal{J}_\gamma(t)U_\gamma(t)\,,$$

which refers the seeingly natural generator $\mathcal{J}_\gamma(t)$ to the initial time.

We do this because

$$\mathcal{K}_\gamma(s) = iU_\gamma^\dagger(s)\frac{\partial U_\gamma(s)}{\partial\gamma}$$

satisfies the temporal differential equation

$$\begin{aligned}\frac{\partial\mathcal{K}_\gamma(s)}{\partial s} &= -\frac{1}{\hbar}U_\gamma^\dagger(s)H(s)\frac{\partial U_\gamma(s)}{\partial\gamma} + \frac{1}{\hbar}U_\gamma^\dagger(s)\underbrace{\frac{\partial}{\partial\gamma}[H(s)U_\gamma(s)]}_{=\hbar h U_\gamma(s) + H(s)\frac{\partial U_\gamma(s)}{\partial\gamma}} \\ &= U_\gamma^\dagger(s)hU_\gamma(s)\,,\end{aligned}$$

which, since $\mathcal{K}_\gamma(0) = 0$, has the straightforward solution

$$\mathcal{K}_\gamma(t) = \int_0^t ds\, U_\gamma^\dagger(s)hU_\gamma(s) \equiv t\overline{h}\,.$$

Proof of QCRB. Quantum mechanics

$$F(\gamma)=\int d\gamma_{\rm est}\,\frac{1}{p(\gamma_{\rm est}|\gamma)}\left(\frac{\partial p(\gamma_{\rm est}|\gamma)}{\partial\gamma}\right)^2$$

$$\frac{\partial p(\gamma_{\rm est}|\gamma)}{\partial\gamma}=\mathrm{tr}\left(E_{\gamma_{\rm est}}\frac{\partial\rho_\gamma(t)}{\partial\gamma}\right)=\mathrm{Re}[\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma(t))]$$

$$\begin{aligned}
\left(\frac{\partial p(\gamma_{\rm est}|\gamma)}{\partial\gamma}\right)^2 &= \Big(\mathrm{Re}[\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma(t))]\Big)^2\\
&\le |\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma)\,|^2\\
&= \left|\mathrm{tr}\left(\sqrt{\rho_\gamma}\sqrt{E_{\gamma_{\rm est}}}\sqrt{E_{\gamma_{\rm est}}}\mathcal{L}_\gamma\sqrt{\rho_\gamma}\right)\right|^2\\
\text{Schwarz inequality}\quad &\le \mathrm{tr}\big(\sqrt{\rho_\gamma}E_{\gamma_{\rm est}}\sqrt{\rho_\gamma}\big)\,\mathrm{tr}\big(\sqrt{\rho_\gamma}\mathcal{L}_\gamma E_{\gamma_{\rm est}}\mathcal{L}_\gamma\sqrt{\rho_\gamma}\big)\\
&= \mathrm{tr}(E_{\gamma_{\rm est}}\rho_\gamma)\,\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma\mathcal{L}_\gamma)\\
&= p(\gamma_{\rm est}|\gamma)\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma\mathcal{L}_\gamma)
\end{aligned}$$

$$F(\gamma)\le\int d\gamma_{\rm est}\,\mathrm{tr}(E_{\gamma_{\rm est}}\mathcal{L}_\gamma\rho_\gamma\mathcal{L}_\gamma)=\mathrm{tr}(\mathcal{L}_\gamma^2\rho_\gamma(t))\equiv Q(\gamma)$$

QCRB

Quantum Fisher information $Q(\gamma)=\langle(\Delta\mathcal{L}_\gamma)^2\rangle$

Proof of QCRB. Summary before optimization

$$\langle(\delta\gamma)^2\rangle = \geq \frac{1}{F(\gamma)} + \langle\delta\gamma\rangle^2 \quad \geq \quad \frac{1}{F(\gamma)}$$

$$\geq \quad \frac{1}{Q(\gamma)} = \frac{1}{\langle(\Delta\mathcal{L}_\gamma)^2\rangle}$$

Proof of QCRB. Optimizing the measurement

$$F(\gamma) \le \mathrm{tr}(\mathcal{L}_\gamma^2 \rho_\gamma(t)) \equiv Q(\gamma) \quad \textbf{QCRB}$$

Saturating the QCRB requires using a measurement that saturates the two inequalities that separate $F(\gamma)$ from $Q(\gamma)$:

$$\mathrm{tr}(E_{\gamma_{\mathrm{est}}} \mathcal{L}_\gamma \rho_\gamma(t)) \text{ real},$$
$$\sqrt{E_{\gamma_{\mathrm{est}}}} \mathcal{L}_\gamma \sqrt{\rho_\gamma} \propto \sqrt{E_{\gamma_{\mathrm{est}}}} \sqrt{\rho_\gamma}\,.$$

Although this looks like a disaster, it is easy to see that measuring the symmetric logarithmic derivative, i.e., measuring in the eigenbasis of $\mathcal{L}_\gamma$,

$$\mathcal{L}_\gamma = \sum_j \lambda_j |j\rangle\langle j|\,,$$

with

$$E_j = |j\rangle\langle j|\,,$$

does the job, because

$$\mathrm{tr}(E_j \mathcal{L}_\gamma \rho_\gamma(t)) = \lambda_j \langle j|\rho_\gamma(t)|j\rangle \text{ is real}$$

and

$$\sqrt{E_j} \mathcal{L}_\gamma = \lambda_j |j\rangle\langle j| = \lambda_j \sqrt{E_j}\,.$$

It is up to you to figure out how to turn the measurement outcome j into an estimate of γ. Moreover, even though this is an optimal measurement, it might not be the only optimal measurent.

Proof of QCRB. Optimizing the state

$$F(\gamma) \le \mathrm{tr}(\mathcal{L}_\gamma^2 \rho_\gamma(t)) \equiv Q(\gamma)$$

$$\frac{\partial \rho_\gamma(t)}{\partial \gamma} = \frac{1}{2}[\mathcal{L}_\gamma \rho_\gamma(t) + \rho_\gamma(t)\mathcal{L}_\gamma] = -iU_\gamma(t)[K_\gamma, \rho_0]U_\gamma^\dagger(t)$$

$$K_\gamma = iU_\gamma^\dagger(t)\frac{\partial U_\gamma(t)}{\partial \gamma} = \int_0^t ds\, U_\gamma^\dagger(s) h U_\gamma(s) \equiv t\overline{h}$$

Specialize to pure-state input: differentiating $\rho_\gamma = \rho_\gamma^2$ gives

$$\mathcal{L}_\gamma = 2\frac{\partial \rho_\gamma(t)}{\partial \gamma} = -2iU_\gamma(t)[K_\gamma, \rho_0]U_\gamma^\dagger(t)\ .$$

$$\begin{aligned}
Q(\gamma) &= -4\mathrm{tr}\big([K_\gamma, \rho_0]^2 U_\gamma^\dagger(t)\rho_\gamma(t)U_\gamma(t)\big) \\
&= -4\mathrm{tr}\big([K_\gamma, \rho_0]^2 \rho_0\big) \\
&= 4\big[\mathrm{tr}(K_\gamma^2 \rho_0) - \mathrm{tr}(K_\gamma \rho_0)^2\big] \\
&= 4\langle(\Delta K_\gamma)^2\rangle \\
&= 4t^2\langle(\Delta \overline{h})^2\rangle
\end{aligned}$$

If $\tilde{H} = 0$, $U_\gamma(t) = e^{-i\gamma h t}$, $K_\gamma = th$, $\overline{h} = h$, and $\mathcal{L}_\gamma = -2it[h, \rho_\gamma(t)]$.

Proof of QCRB. Optimizing the state

$$\langle(\delta\gamma)^2\rangle \geq \frac{1}{F(\gamma)} \geq \frac{1}{Q(\gamma)}$$

$$Q(\gamma) = 4t^2\langle(\Delta\overline{h})^2\rangle$$

$$\overline{h} = \frac{1}{t}\int_0^t ds\, U_\gamma^\dagger(s) h U_\gamma(s)$$

$$\frac{1}{\langle(\delta\gamma)^2\rangle^{1/2}} \leq \sqrt{F(\gamma)} \leq \sqrt{Q(\gamma)} = 2t\langle(\Delta\overline{h})^2\rangle^{1/2} \leq t||\overline{h}|| \leq t||h||$$

$||h||$ is the seminorm of h, i.e., the difference between largest and smallest eigenvalues of h.

Optimal state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\Lambda\rangle + |\lambda\rangle)$$

$|\Lambda\rangle$ is the largest-eigenvalue eigenstate of $\overline{h}$ or h, and $|\lambda\rangle$ is the smallest-eigenvalue eigenstate.

Triangle inequality:

$$||h_1 + h_2|| \leq ||h_1|| + ||h_2||$$

The bassackwardness of the QCRB. The final detail

$$\langle(\delta\gamma)^2\rangle^{1/2} \geq \frac{1}{\sqrt{F(\gamma)}} \geq \frac{1}{\sqrt{Q(\gamma)}} = \frac{1}{2t\langle(\Delta\overline{h})^2\rangle^{1/2}} \geq \frac{1}{t||\overline{h}||} \geq \frac{1}{t||h||}$$

Optimal state $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\Lambda\rangle + |\lambda\rangle)$$

$|\Lambda\rangle$ is the largest-eigenvalue eigenstate of $\overline{h}$ or h, and $|\lambda\rangle$ is the smallest-eigenvalue eigenstate.

The bassackwardness of the QCRB is that to make the most sensitive measurement possible, you are instructed to make an observable, the generator $\overline{h}$ or h, as noisy as possible.

What the QCRB only whispers to you—and most people don't even hear the whisper—is that for pure states, there is a very quiet (conjugate) observable, the symmetric logarithmic derivative $\mathcal{L}_\gamma$, and you are instructed to measure that.

When $\tilde{H}(t) = 0$, with the optimal input state $|\Psi_0\rangle$,

$$\mathcal{L}_\gamma = -2it[h, e^{-i\gamma ht}|\Psi_0\rangle\langle\Psi_0|e^{i\gamma ht}]\,,$$

and this is the obvious observable to measure.

The bassackwardness of the QCRB. The final detail

When $\tilde{H}(t)=0$, with the optimal input state $|\Psi_0\rangle = (|\Lambda\rangle + |\lambda\rangle)/\sqrt{2}$,

$$\mathcal{L}_\gamma = -2it[h, e^{-i\gamma ht}|\Psi_0\rangle\langle\Psi_0|e^{i\gamma ht}]\,,$$

and this is the obvious observable to measure *because this becomes a qubit problem.*

$$\begin{aligned} h &= \Lambda|\Lambda\rangle\langle\Lambda| + \lambda|\lambda\rangle\langle\lambda| + \text{(irrelevant orthogonal stuff)} \\ &= \frac{1}{2}(\Lambda+\lambda)I + \frac{1}{2}(\Lambda-\lambda)Z + \text{(irrelevant orthogonal stuff)} \end{aligned}$$

$$e^{-i\gamma ht}|\Psi_0\rangle = \frac{1}{\sqrt{2}}(e^{-i\gamma\Lambda t}|\Lambda\rangle + e^{-i\gamma\lambda t}|\lambda\rangle) \quad +1 \text{ eigenstate of qubit Pauli } X_{\text{rot}}$$

$$e^{-i\gamma ht}|\Psi_0\rangle\langle\Psi_0|e^{i\gamma ht} = \frac{1}{2}\Big(\underbrace{|\Lambda\rangle\langle\Lambda| + |\lambda\rangle\langle\lambda|}_{\text{Pauli } I} + \underbrace{e^{-i\gamma(\Lambda-\lambda)t}|\Lambda\rangle\langle\lambda| + e^{i\gamma(\Lambda-\lambda)t}|\lambda\rangle\langle\Lambda|}_{\text{Pauli } X_{\text{rot}}}\Big)$$

$$\begin{aligned} \mathcal{L}_\gamma &= -2it[h, e^{-i\gamma ht}|\Psi_0\rangle\langle\Psi_0|e^{i\gamma ht}] \\ &= (\Lambda-\lambda)t\Big[\underbrace{-i\Big(e^{-i\gamma(\Lambda-\lambda)t}|\Lambda\rangle\langle\lambda| - e^{i\gamma(\Lambda-\lambda)t}|\lambda\rangle\langle\Lambda|\Big)}_{=\text{ Pauli } Y_{\text{rot}}}\Big] \end{aligned}$$

The End