One Hundred Years After Heisenberg: Discovering the World of Simultaneous Measurements of Noncommuting Observables

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> "Well, why not say that all the things which should be handled in theory are just those things which we also can hope to observe somehow." … I remember that when I first saw Einstein I had a talk with him about this. … [H]e said, ``That may be so, but still it's the wrong principle in philosophy." And he explained that it is the theory finally which decides what can be observed and what can not and, therefore, one cannot before the theory, know what is observable and what not.
>
> Werner Heisenberg, recalling a conversation with Einstein in 1926,
> interviewed by Thomas S. Kuhn, February 15, 1963

This work was carried out with Christopher S. Jackson, whose genius and vision inform every aspect.

A brief glimpse into a whole new world

Moo Stack and the Villians of Ure
Eshaness, Shetland

A brief glimpse into a whole new world

Any set of Hermitian observables,

$$\vec{X} = \{X_1, \ldots, X_n\},$$

can be measured (differential) weakly and simultaneously in an infinitesimal increment dt, without regard to commutators.

Examples

Single observable X

Position Q and momentum P

Three components of angular momentum, J_x, J_y, and J_z

Two components of angular momentum, J_x and J_y

Concatenating these differential weak measurements continuously should tell one what it means to measure the same observables strongly and simultaneously. And so it does, except that it also leads to . . .

A magic carpet ride
Into a whole new world.
A new, fantastic point of view,
A thrilling chase,
A wondrous space,
And now we bring this whole new world to you.

Big-time apologies to *Aladdin*

A brief glimpse into a whole new world

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Big-time apologies to *Aladdin*

The details in three long papers
C. S. Jackson and C. M. Caves

"Simultaneous measurements of noncommuting observables: Positive transformations and instrumental Lie groups," Entropy 25, 1254 (2023). (a.k.a. 1-2-3), https://doi.org/10.3390/e25091254

"Simultaneous momentum and position measurement and the instrumental Weyl-Heisenberg group," Entropy 25, 1221 (2023). (a.k.a. SPQM), https://doi.org/10.3390/e25081221

"How to perform the coherent measurement of a curved phase space by continuous isotropic measurement. I. Spin and the Kraus-operator geometry of SL(2,C)," Quantum 7, 1085 (2023), arXiv:2107.12396v3. (a.k.a ISM), https://doi.org/10.22331/q-2023-08-16-1085

A brief glimpse into a whole new world

- Any set of observables can be measured simultaneously if measured differential weakly. Commutators can be disregarded for *differential weak measurements*.
- Differential weak measurements define a *fundamental incremental Kraus operator*, a *differential positive transformation,* which is the positive-operator analogue of an infinitesimal unitary transformation and equally fundamental.
- Instrument (Kraus-operator) evolution is *autonomous*, *temporal,* and *stochastic.*
- *Instrument Manifold Program.* The instrument evolution occurs on the manifold of an *instrumental Lie group*, which is generated by the measured observables.
- Motion of the Kraus operators on the instrumental Lie group is described using the three faces of the *stochastic trinity:* Wiener path integrals, stochastic differential equations, and a diffusion equation for a *Kraus-operator distribution function.*
- *Universal instruments.* The instrumental Lie group is generated universally, detached from and independent of Hilbert space.
- *Principal instruments* (e.g., measure position and momentum or three components of spin) have a low-dimensional universal instrumental group: they limit to coherent-state POVMs—*collapse within irrep*—and thus define a phase space, which is connected to the identity across a symmetric space. These instruments are special and universal (pre-quantum) and structure any Hilbert space in which they are represented; *principal instruments are what Heisenberg and Einstein meant when they talked about identifying what is observable.*
- *Chaotic instruments* (e.g., measure two components of spin) have an infinite-dimensional universal instrumental group: these are generic, evolve chaotically, and have no universal limiting strong measurement.

Not about making better measurements, this talk *is* about thinking of measurements in a new way, which places them at the foundation of quantum mechanics.

100 years of quantum measurements

Pinnacles National Park
Central California

100 years of quantum measurements.
The founding (1925–32)

Matrix mechanics, commutators, and uncertainty principle
Wave mechanics (the Schrödinger equation)
Linear algebra of square-integrable functions
Dirac-Jordan transformation theory
Born probability rule

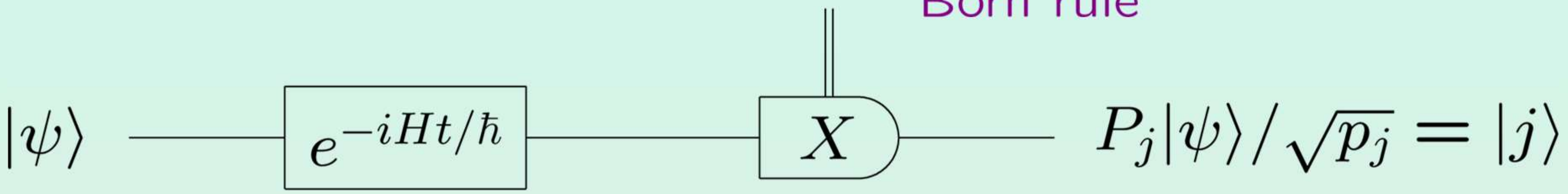
von Neumann's synthesis: inner products and Hilbert space, unitary transformations (Hamiltonian dynamics), and measurements of Hermitian observables

von Neumann measurement of Hermitian observable

$$X = \sum_j \lambda_j |j\rangle\langle j| = \sum_j \lambda_j P_j$$

$$j,\ p_j = |\langle j|\psi\rangle|^2 = \langle\psi|P_j|\psi\rangle$$

Born rule



State vector (pure state)

unitary (Hamiltonian) dynamics

Collapse

Temporal: (continuous) *process*

Autonomous: independent of state

Temporal	Autonomous	Transformation	Group
No	Yes	Yes	No

Physics by fiat: no simultaneous measurements of noncommuting observables

Drop $|\psi\rangle$ from description.

100 years of quantum measurements.
The desert (1932–60)

Quantum measurement theory withered under the desert sun, whereas the unitary side of quantum mechanics thrived with constant and well-deserved nurturing.

Everybody used the Born rule, though how to interpret its probabilities and the quantum state remains a source of discussion and debate today. Nobody used and next to nobody bought von Neumann's collapse because there were no repeated measurements on the same system.

All measurements were actually von Neumann's indirect measurements and analyzed using the Born rule without using von Neumann measurements of Hermitian observables.

Mathematical developments

Unitary Lie groups: symmetry groups and representation theory
Functional and harmonic analysis
Functional (path) integration
Transformation groups
Differential geometry of complex Lie groups
Measures and probability theory, stochastic processes, and stochastic calculus

First three and a bit of the fourth fell on fertile soil in the unitary sector of quantum mechanics and quantum field theory.

None of this got into (or was needed in) the desiccated quantum measurement theory.

We use all these.

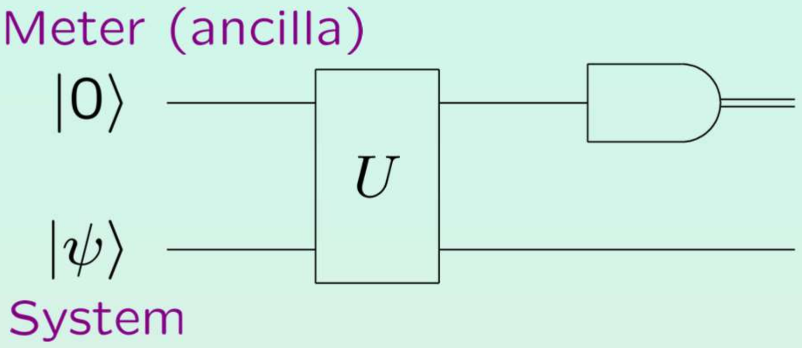
100 years of quantum measurements.
Generalized measurement theory (1960–85)

Overcomplete-basis measurements (measurements of noncommuting observables, coherent states, heterodyne)

Hint of repeated measurements

Generalized measurement theory. Taking advantage of von Neumann's indirect measurements

Wigner
Davies
Ludwig
Kraus



Born rule

$$j,\ p_j = \mathrm{tr}_S(K_j|\psi\rangle\langle\psi|K_j^\dagger) = \mathrm{tr}_S(E_j|\psi\rangle\langle\psi|)$$

$$\langle j|U|0\rangle|\psi\rangle/\sqrt{p_j} = K_j|\psi\rangle/\sqrt{p_j}$$

"Gentler" collapse

(Kraus operator) $= K_j = \langle j|U|0\rangle$

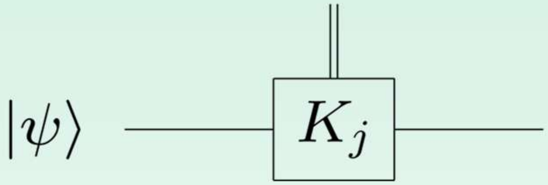
(POVM element) $= E_j = K_j^\dagger K_j \geq 0$

(Completeness): $\sum_j E_j = 1$

Positive

$$\Longleftrightarrow$$

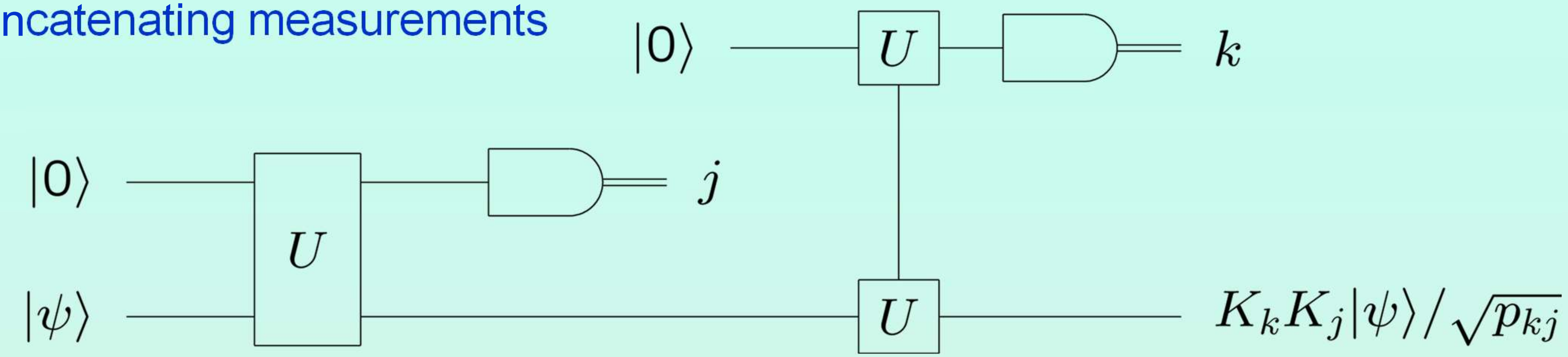
$$j,\ p_j = \langle\psi|E_j|\psi\rangle$$



$$K_j|\psi\rangle/\sqrt{p_j}$$

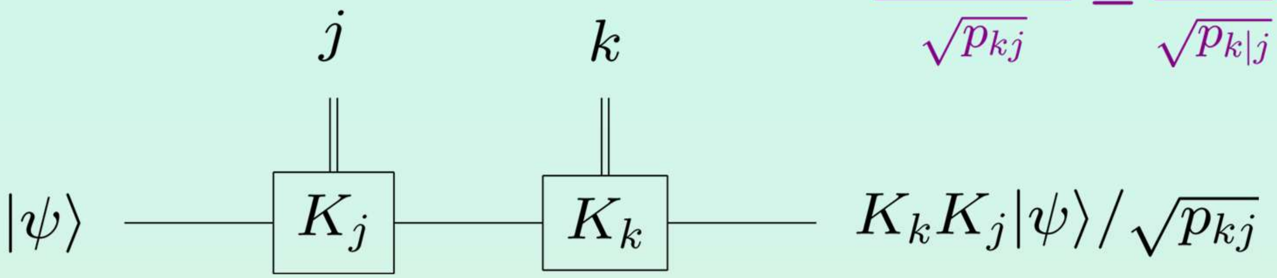
100 years of quantum measurements.
Generalized measurement theory (1960–85)

Concatenating measurements



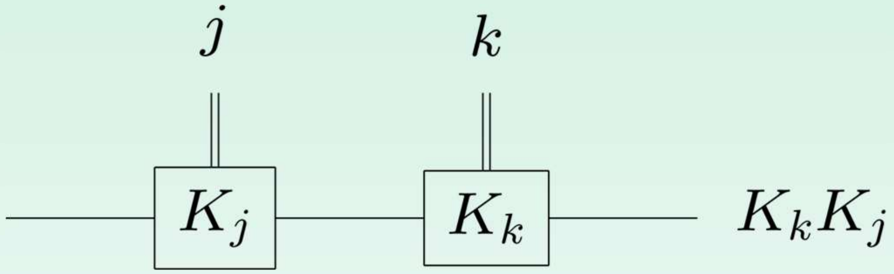
$$\frac{K_kK_j|\psi\rangle}{\sqrt{p_{kj}}} = \frac{K_k}{\sqrt{p_{k|j}}}\frac{K_j}{\sqrt{p_j}}|\psi\rangle$$

$\Longleftrightarrow$



Temporal	Autonomous	Transformation	Group
Gettin' there	?	Yes	No

To normalize or not to normalize?



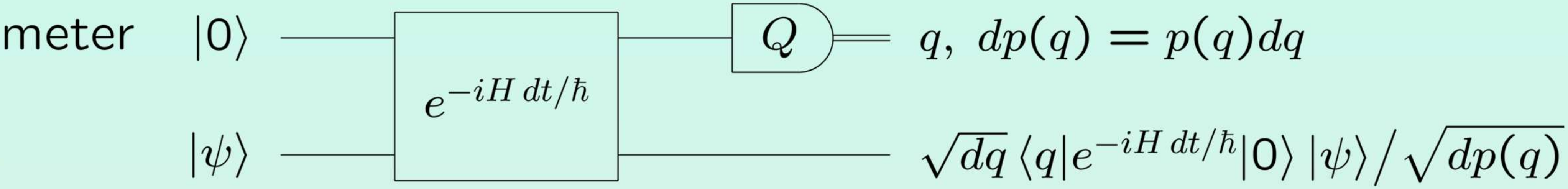
100 years of quantum measurements.
Continuous weak measurements (1980–2010)

We are interested in differential weak measurements: Kraus operators close to the identity.

Differential weak measurement of *X* in increment *dt*

Davies
Barchielli
Carmichael
Milburn
Wiseman
Goetsch/Graham
Doherty
Mabuchi
Jacobs
Brun
Steck

Meter wave function: $\langle q|0\rangle = \sqrt{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-q^2/2\sigma^2}}$



$H\,dt = 2\sqrt{\kappa\,dt}\,\sigma P \otimes X$
Controlled displacement
of meter position Q by X

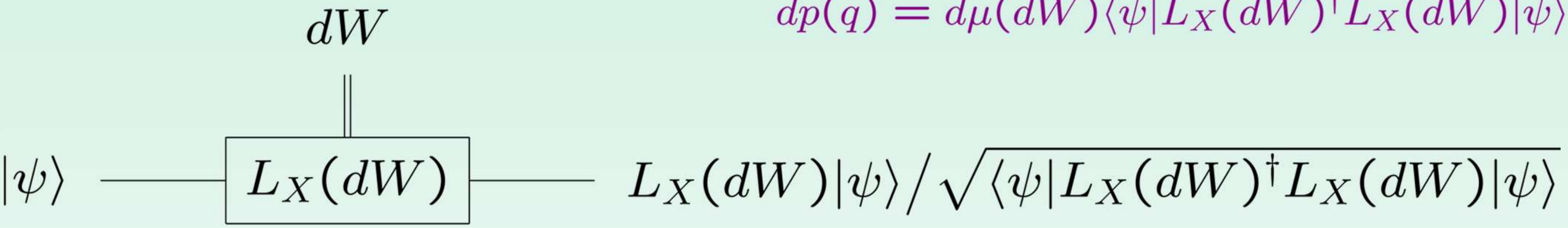
$dW = q\sqrt{dt}/\sigma$ is a *Wiener outcome increment.*
$d\mu(dW) =$ zero-mean Gaussian with $\langle dW^2\rangle = dt$

Kraus operator

$$\sqrt{dq}\,\langle q|e^{-iH\,dt/\hbar}|0\rangle = \sqrt{d\mu(dW)}\;\underbrace{e^{X\sqrt{\kappa}\,dW - X^2\kappa\,dt}}_{=L_X(dW)}$$

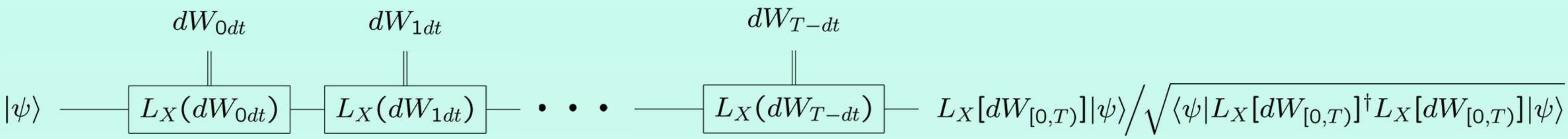
$$dp(q) = d\mu(dW)\langle\psi|L_X(dW)^\dagger L_X(dW)|\psi\rangle$$

Alert: Don't faint at the sight of a dW or, even worse, a d(dW).



100 years of quantum measurements.
Continuous weak measurements (1980–2005)

Concatenating: Continuous, differential weak measurement of *X* over finite time *T*.



$$\text{(incremental Kraus operator)} = L_X(dW_t) = e^{\delta_t} = e^{X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt}$$

$$\text{(forward generator)} = \delta_t \equiv X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt$$

$$\text{(overall Kraus operator)} = L\big[dW_{[0,T)}\big] = \mathcal{T}\prod_{k=0}^{T/dt-1} L_X(dW_{kdt})$$

$$= \mathcal{T}\exp\left(\int_0^{T-dt} X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt\right)$$

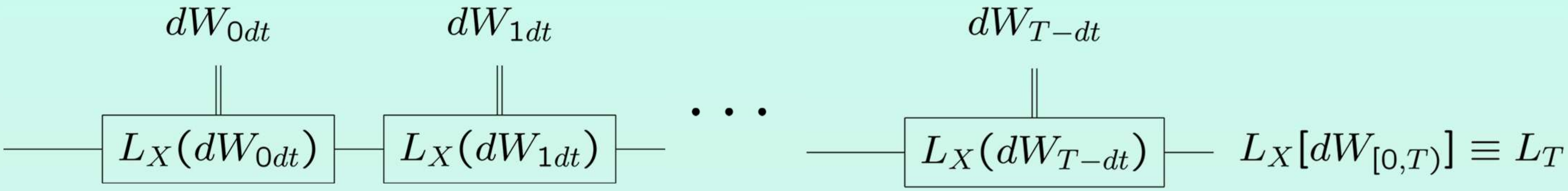
Temporal	Autonomous	Transformation	Group
Yes	?	Yes	?

Normalizing at each increment gives a stochastic master equation for an evolving state. Not normalizing, sometimes called a linear quantum trajectory, gives autonomous instrument evolution and a Lie group.

100 years of quantum measurements.
Continuous weak measurements (1980–2005)

Continuous, differential weak measurement of *X* over finite time *T*.

Instrument evolution



$$L_X(dW_t) = e^{\delta_t} = e^{X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt}\,, \qquad \delta_t \equiv X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt$$

$$L_T \equiv L\big[dW_{[0,T)}\big] = \mathcal{T}\prod_{k=0}^{T/dt-1} L_X(dW_{kdt}) = \mathcal{T}\exp\left(\int_0^{T-dt} X\sqrt{\kappa}\,dW_t - X^2\kappa\,dt\right)$$

Temporal	Autonomous	Transformation	Group
Yes	Yes	Yes	Yes

Continuous measurements of a single observable are trivial because everything commutes (time ordering is irrelevant; irreps are 1D). They limit to a strong measurement that is a von Neumann measurement (standard collapse between irreps).

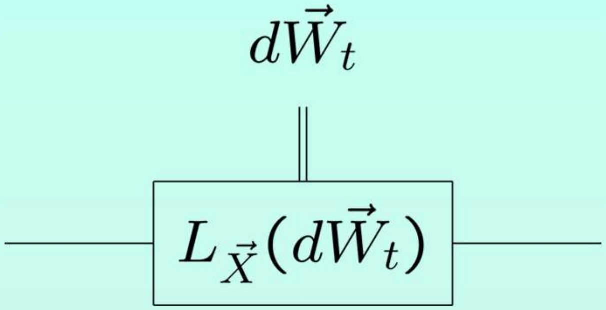
Variegated fairy wren
Oxley Common, Brisbane

Red-backed fairy wren
Oxley Common, Brisbane

Instrument manifold program

Western diamondback rattlesnake
My front yard, Sandia Heights

Instrument manifold program



The incremental Kraus operator for a differential weak measurement of n (noncommuting) observables $\vec{X} = \{X_1, \ldots, X_n\}$, beginning at time t for an increment dt, is

$$\sqrt{d\mu(d\vec{W}_t)}\, L_{\vec{X}}(d\vec{W}_t) = \sqrt{d(dW_t^1)\cdots d(dW_t^n)\frac{e^{-d\vec{W}_t\cdot d\vec{W}_t/2dt}}{(2\pi dt)^{n/2}}}\; e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t-\vec{X}^2\kappa\,dt},$$

Commutators can be disregarded for *differential weak measurements*.

Exponentials for different observables can be combined into a single exponential at order *dt*—commutators ignored—because the Wiener outcome increments are uncorrelated.

$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t}, \qquad \delta_t \equiv \vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt = (\text{forward generator})$$

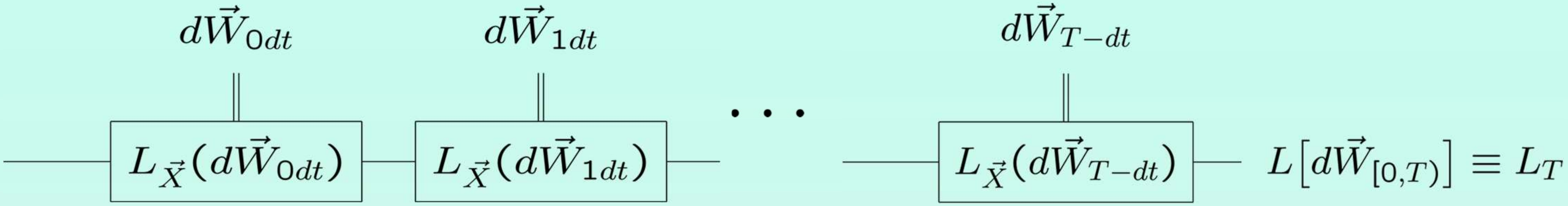
Differential weak measurements define a *fundamental incremental Kraus operator*, a *differential positive transformation,* which is fundamental in the same way as infinitesimal unitary transformations.

$$\vec{X}\cdot d\vec{W}_t = \sum_\mu X_\mu dW_t^\mu, \qquad dW_t^\mu = (\text{Wiener outcome increment for } X_\mu)$$

$$\vec{X}^2 = \vec{X}\cdot\vec{X} = \sum_\mu X_\mu^2 = (\text{quadratic term}).$$

Instrument manifold program

Instrument evolution: “piling up” Kraus operators



$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t} = e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt}$$

The differential positive transformations “pile up” as successive incremental measurements are performed; at time T the Kraus operator is

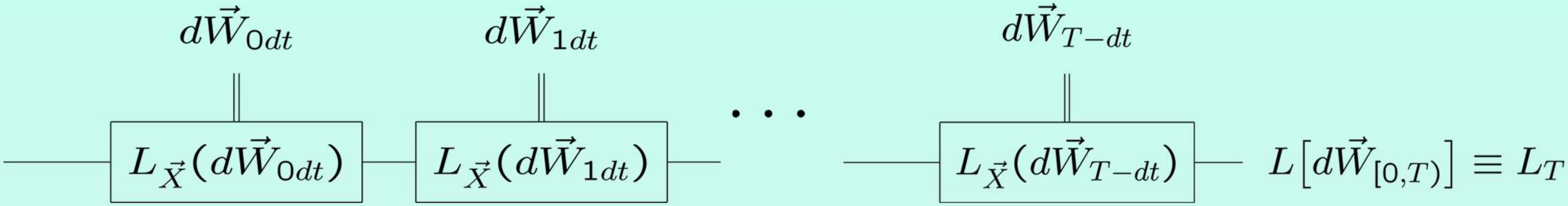
$$L_T = L\big[d\vec{W}_{[0,T)}\big] = \mathcal{T}\exp\left(\int_0^{T-dt}\delta_t\right) = \mathcal{T}\exp\left(\int_0^{T-dt}\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt\right),$$

where $d\vec{W}_{[0,T)}$ is the Wiener outcome path and $\mathcal{T}$ denotes a time-ordered exponential—commutators do count for finite T! The measuring *instrument* is the collection of these Kraus operators for all Wiener paths (more precisely, the collection of *instrument elements* $L_T \odot L_T^\dagger$).

Instrument evolution is a *process: autonomous, temporal, and stochastic.*

Instrument manifold program

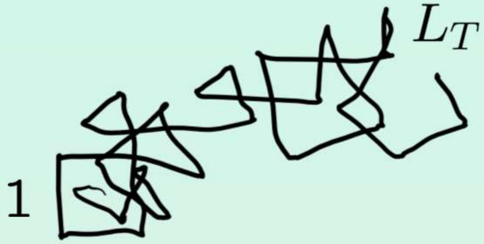
Instrument evolution



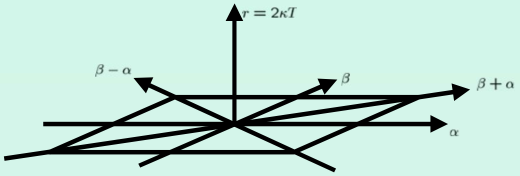
$$L_{\vec{X}}(d\vec{W}_t) = e^{\delta_t} = e^{\vec{X}\cdot\sqrt{\kappa}\,d\vec{W}_t - \vec{X}^2\kappa\,dt}$$

The quantum circuit becomes a stochastic path on the instrumental Lie group manifold.

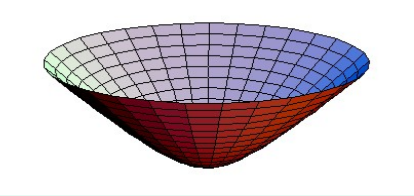
Alert: New thinking; new concepts and techniques.



$L_T = L\big[d\vec{W}_{[0,T)}\big]$



SPQM



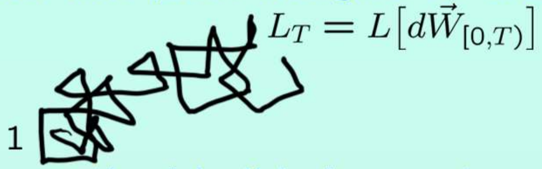
ISM

The Kraus operators $L_T = L\big[d\vec{W}_{[0,T)}\big]$ are elements of an *instrumental Lie group* G, which is the group generated by the measured observables, $\vec{X} = \{X_1, \ldots, X_n\}$, and the *quadratic term*, $\vec{X}^2 = \vec{X}\cdot\vec{X} = \sum_\mu X_\mu^2$. The instrument evolves stochastically in the manifold of the instrumental Lie group, which is the natural setting for the measurement.

Heart of the *Instrument Manifold Program:*
Instrumental Lie groups.

Instrument manifold program. How we do it.

The motion of the Kraus operators on the group manifold G is analyzed using the three faces of the *stochastic trinity*. The overall quantum operation is given by a Wiener-like path integral of the measurement record,



$$\mathcal{Z}_T = \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\; L[d\vec{W}_{[0,T)}] \odot L[d\vec{W}_{[0,T)}]^\dagger\,;$$

it is the solution of a Lindblad equation in which the measured observables are the Lindblad operators. The Kraus-operator paths satisfy a *stochastic differential equation* (SDE),

Maurer-Cartan form (Stratonovich): $dL_t\,L_{t+dt/2}^{-1} = \delta_t = \vec{X}\cdot\sqrt{\kappa}\,d\vec{W} - \vec{X}^2\kappa\,dt\,,$

Modified MC stochastic differential (Itô): $dL_t\,L_t^{-1} - \frac{1}{2}(dL_t\,L_t^{-1})^2 = \delta_t\,.$

Stochastic differential equation

A *Kraus-operator distribution function* (KOD) is defined by a Wiener path integral,

Alert: New thinking; new concepts and techniques.

$$D_T(L) \equiv \int \mathcal{D}\mu[d\vec{W}_{[0,T)}]\; \delta\big(L, L[d\vec{W}_{[0,T)}]\big)\,,$$

Wiener path integral

where the δ-function is defined relative to the Haar measure of G. The KOD evolves according to a *Fokker-Planck-Kolmogorov equation* (FPKE),

$$\frac{1}{\kappa}\frac{\partial D_t(L)}{\partial t} = \Delta[D_t](L)\,, \qquad \Delta \equiv \underleftarrow{\vec{X}}^2 + \frac{1}{2}\sum_\mu \underleftarrow{X_\mu}\,\underleftarrow{X_\mu}\,,$$

Diffusion equation

where the underarrows denote *right-invariant derivatives*,

$$\underleftarrow{X}[f](L) \equiv \frac{d}{dh} f(e^{hX}L)\bigg|_{h=0} = \lim_{h\to 0}\frac{f(e^{hX}L) - f(L)}{h}\,,$$

these being the natural vector fields on the instrumental Lie-group manifold. Notice that $\underleftarrow{\vec{X}}^2$ describes ballistic motion and

$$\nabla^2 \equiv \sum_\mu \underleftarrow{X_\mu}\,\underleftarrow{X_\mu}$$

is a Laplacian that describes diffusion.

Stochastic trinity

Instrument manifold program

The Lie algebra for the instrumental Lie group can be processed using the matrices of a particular representation or processed *universally*, using only the commutators, within what is called the *universal enveloping algebra*. The *universal instrumental Lie group* is called the *universal covering group*; it is detached from Hilbert space.

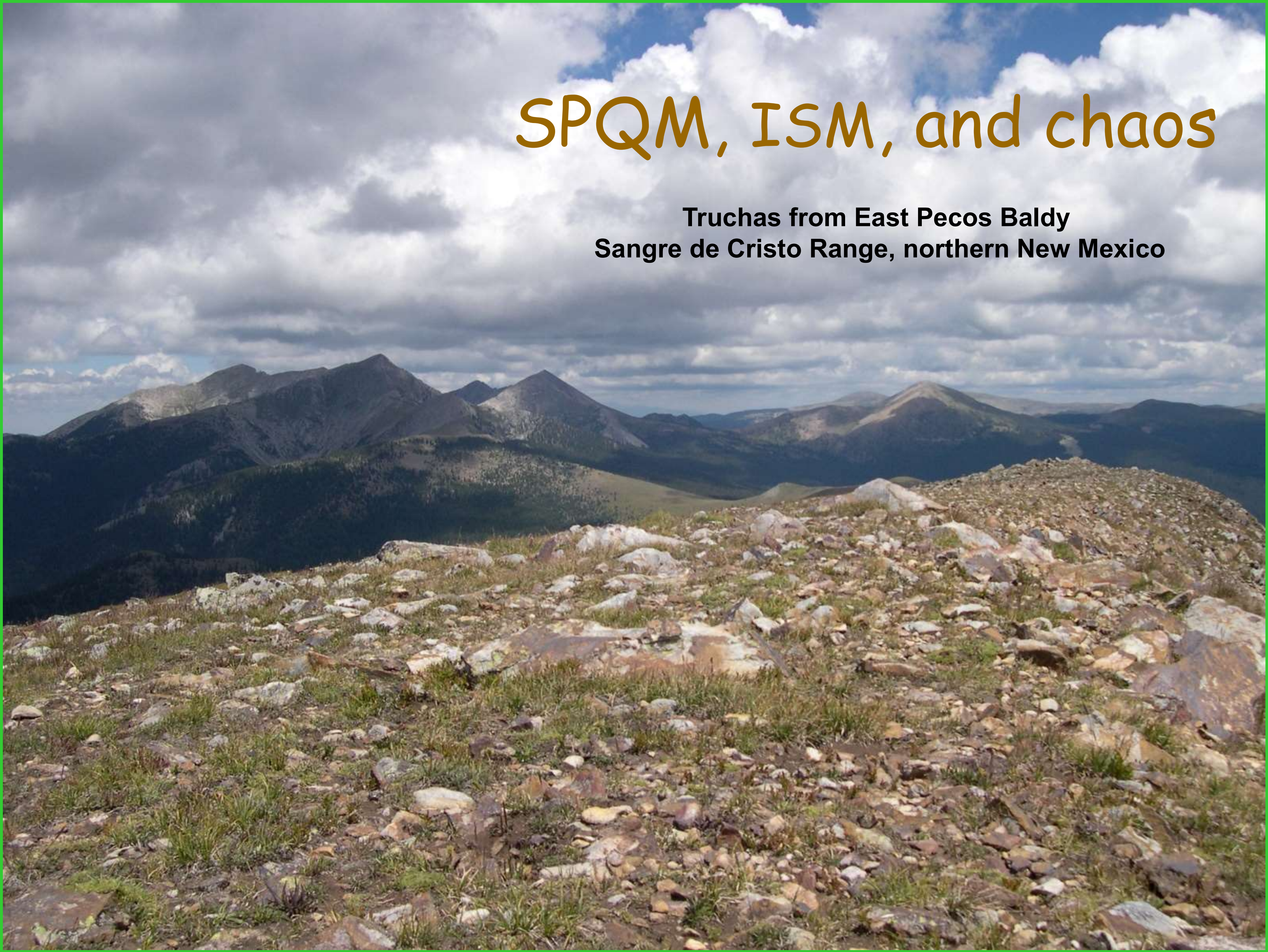
Universal instruments detached from Hilbert space.

Special measurements, such as simultaneous momentum and position measurement (SPQM) and isotropic measurement of the three spin components (ISM), have a low-dimensional universal instrumental group; the instrument's POVM approaches a boundary of coherent states at late times, which is the strong measurement of the observables (we call this collapse within irrep). These instruments we call *principal instruments*. They connect classical phase space to the identity across a (type-IV) *symmetric space*.

Generic measurements—e.g., two components of spin, two squeezing symplectic generators—have an infinite-dimensional universal instrumental group. The instrument's evolution is chaotic; there is no universal (i.e., representation-independent) strong measurement of the observables. These instruments we call *chaotic instruments*.

Principal universal instruments. Very special. Pre-quantum.
Coherent-state POVMs and phase space; collapse within irrep; phase space connected to the identity across a symmetric space.
Chaotic universal instruments. Generic.
Chaotic evolution and no limiting universal strong measurement.

SPQM, ISM, and chaos



Truchas from East Pecos Baldy
Sangre de Cristo Range, northern New Mexico

Simultaneous momentum and position measurement (SPQM): A principal instrument

Measure position Q and momentum P.
Commutator: $[Q,P]=i1$
Quadratic term: $Q^2+P^2=2H_\circ$

7D instrumental Lie algebra: span$\{1, i1, iQ, -iP, Q, P, H_\circ\}$

7D instrumental Lie group, the *Instrumental Weyl-Heisenberg Group*,
$\mathrm{IWH} = \mathbb{C}\mathrm{WH} \rtimes e^{\mathbb{R}H_\circ}$,

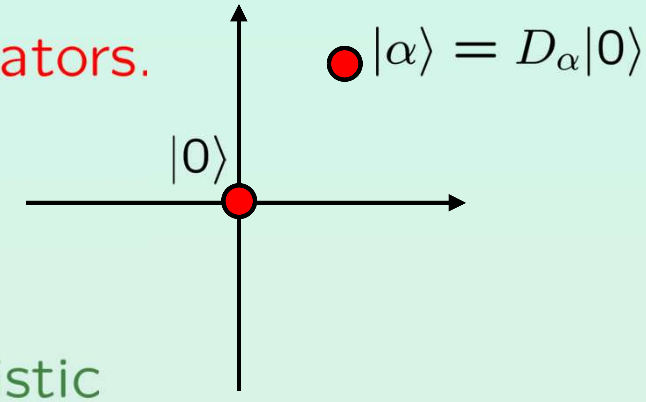
Coördinate manifold with Cartan-like decomposition,
a group-theoretic singular-value decomposition,

Kraus operator $\quad L = \big(D_\beta e^{i1\phi}\big)e^{-H_\circ r - 1\ell}D_\alpha^\dagger$

D_β and D_α are phase-plane displacement operators.

POVM element $\quad E = L^\dagger L = e^{-2\ell\,1}D_\alpha e^{-H_\circ 2r}D_\alpha^\dagger$

Base manifold of positive transformations



$(\ell,\phi) =$ (center normalization and phase)

$r =$ (ruler/purity), $\quad dr = 2\kappa\,dt\,, \quad r = 2\kappa t$ is ballistic

$\beta =$ (post-measurement phase-plane)

$\alpha =$ (POVM phase-plane)

$r = 0: \quad E = L^\dagger L = e^{-2\ell}1$

$r = 2\kappa T: \quad E = L^\dagger L = e^{-2\ell}D_\alpha e^{-H_\circ 2r}D_\alpha^\dagger$

$r \to \infty: \quad E = L^\dagger L = e^{-2\ell}|\alpha\rangle\langle\alpha|$

Write the SDEs and FPKE
in these coördinates.

SPQM: A principal instrument

Kraus operator $L = \left(D_\beta e^{i1\phi}\right) e^{-H_\circ r - 1\ell} D_\alpha^\dagger$

POVM element $E = L^\dagger L = e^{-2\ell 1} D_\alpha e^{-H_\circ 2r} D_\alpha^\dagger$

$(\ell, \phi) =$ (center normalization and phase)

$r =$ (ruler/purity), $dr = 2\kappa\, dt$, $r = 2\kappa t$ is ballistic

$\beta =$ (post-measurement phase-plane)

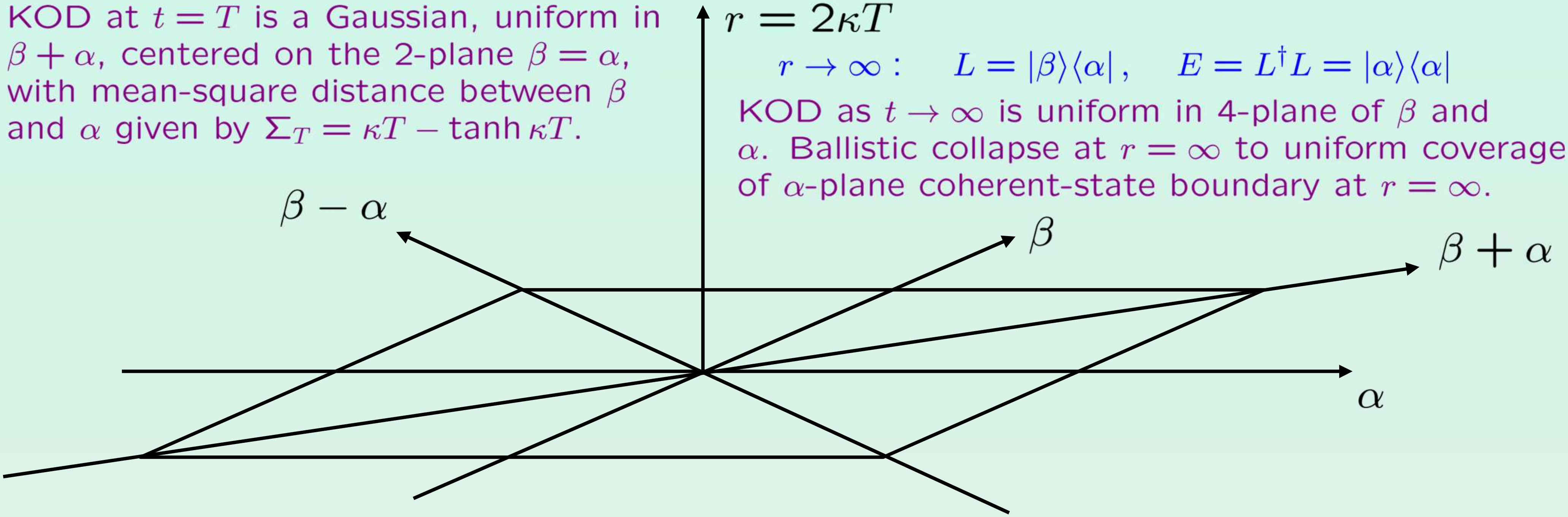
$\alpha =$ (POVM phase-plane)

Work in the coset of the center $Z = \left\{e^{1(-\ell + i\phi)}\right\}$;
i.e., identify points having different center coördinates ℓ and ϕ.

$r = 2\kappa T$: $L = D_\beta e^{-H_\circ 2\kappa T} D_\alpha^\dagger$

$E = L^\dagger L = D_\alpha e^{-H_\circ 4\kappa T} D_\alpha^\dagger$

KOD at $t = T$ is a Gaussian, uniform in $\beta + \alpha$, centered on the 2-plane $\beta = \alpha$, with mean-square distance between β and α given by $\Sigma_T = \kappa T - \tanh \kappa T$.



$r \to \infty$: $L = |\beta\rangle\langle\alpha|$, $E = L^\dagger L = |\alpha\rangle\langle\alpha|$

KOD as $t \to \infty$ is uniform in 4-plane of β and α. Ballistic collapse at $r = \infty$ to uniform coverage of α-plane coherent-state boundary at $r = \infty$.

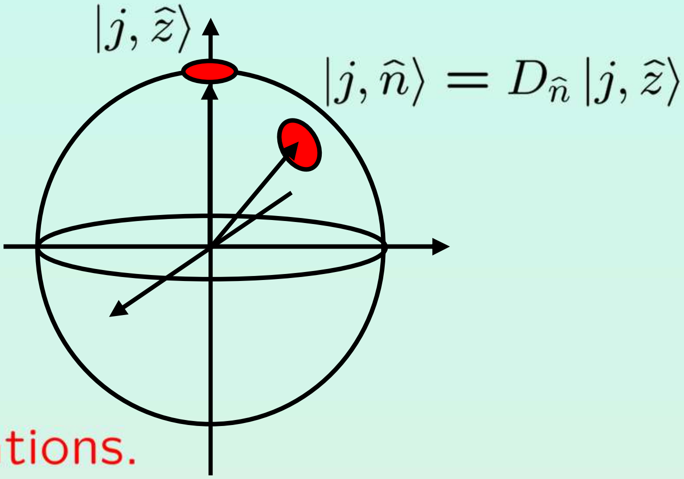
$r = 0$: $L = D_\beta D_\alpha^\dagger = D_{\beta-\alpha}$, $E = L^\dagger L = 1$

KOD at $t = 0$ is a uniform distribution on the 2-plane $\beta = \alpha$ of identity operators.

Isotropic spin measurement (ISM): A principal instrument

Measure J_x, J_y, and J_z.
Commutators: $[J_\mu, J_\nu] = i\epsilon_{\mu\nu\eta}J_\eta$
Quadratic term: $J_x^2 + J_y^2 + J_z^2 = \vec{J}^2 =$ Casimir invariant $= 1_j j(j+1)$

7D instrumental Lie algebra: span$\{-iJ_x, -iJ_y, -iJ_z, J_x, J_y, J_z, \vec{J}^2\}$
7D instrumental Lie group, the *Instrumental Spin Group*, ISpin(3) $=$ SL$(2,\mathbb{C}) \times e^{\mathbb{R}\vec{J}^2}$



Coördinate manifold with Cartan decomposition,
group-theoretic singular-value decomposition,

Kraus operator $\quad L = (D_{\hat{m}}e^{-iJ_z\psi})e^{-\vec{J}^2\ell + J_z a}D_{\hat{n}}^\dagger$

$D_{\hat{m}}$ and $D_{\hat{n}}$ are spherical displacement operators.

POVM element $\quad E = L^\dagger L = e^{-\vec{J}^2 2\ell}D_{\hat{n}}e^{J_z 2a}D_{\hat{n}}^\dagger$

Base manifold (symmetric space) of positive transformations.

$\ell =$ (center normalization), $\quad d\ell = \kappa\, dt\,,$

$\ell = \kappa t$ is ballistic

$\psi =$ (geodesic curvature between past and future)

$a =$ (radial/purity), $\quad da_t = \kappa\, dt \coth a_t + \sqrt{\kappa}\, dY_t^z\,,$

a is ballistic and diffusive

$\hat{m} =$ (post-measurement Bloch sphere)

$\hat{n} =$ (POVM Bloch sphere)

Write the SDE and FPKE
in these coördinates.

$a = 0: \; E = L^\dagger L = e^{-\vec{J}^2 2\kappa t}$

$a \to \infty: \; E = L^\dagger L \propto e^{-\vec{J}^2 2\kappa t}|j,\hat{n}\rangle\langle j,\hat{n}|$

ISM: A principal instrument

Coördinate manifold with Cartan decomposition, group-theoretic singular-value decomposition,

Kraus operator $L = (D_{\hat{m}} e^{-iJ_z\psi}) e^{-\vec{J}^2\ell + J_z a} D_{\hat{n}}^\dagger$

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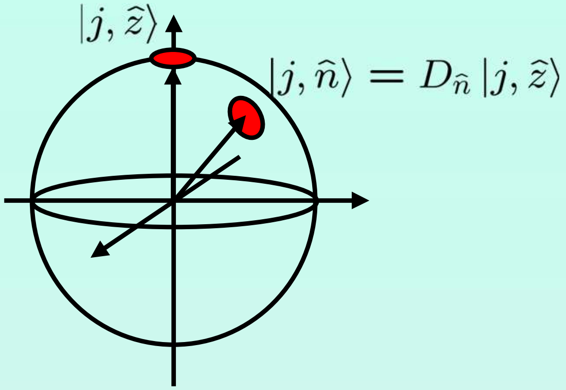

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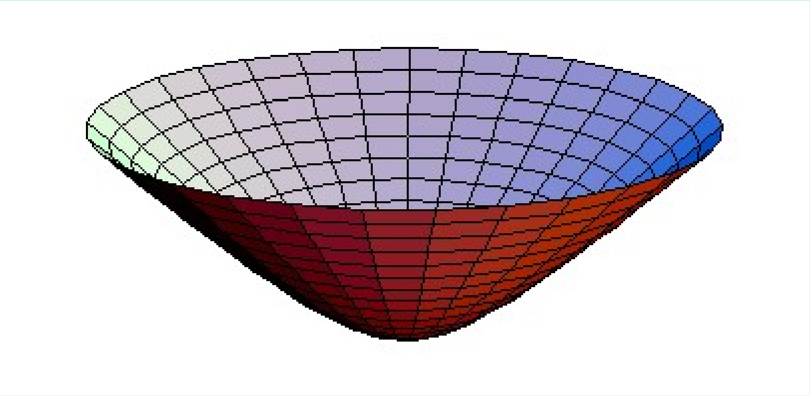
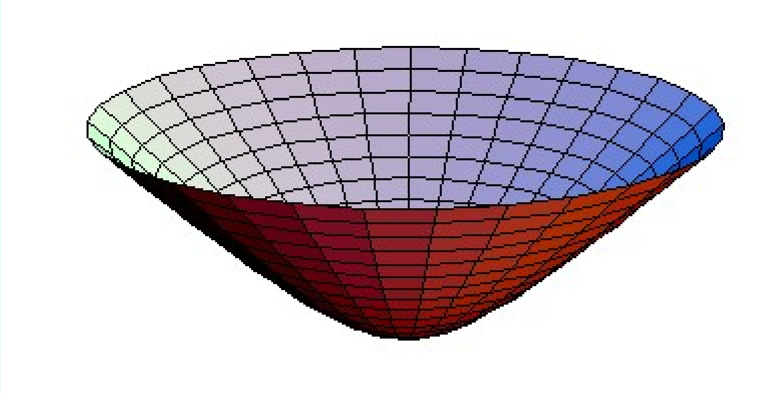


The base manifold of positive transformations (POVM elements) is a symmetric space, in this case a 3-hyperboloid of constant negative curvature, coördinated by a and $\hat{n}$.

To get the induced geometry right, one regards the hyperboloid as embedded in Minkowski space, not Euclidean space.

ISM: A principal instrument



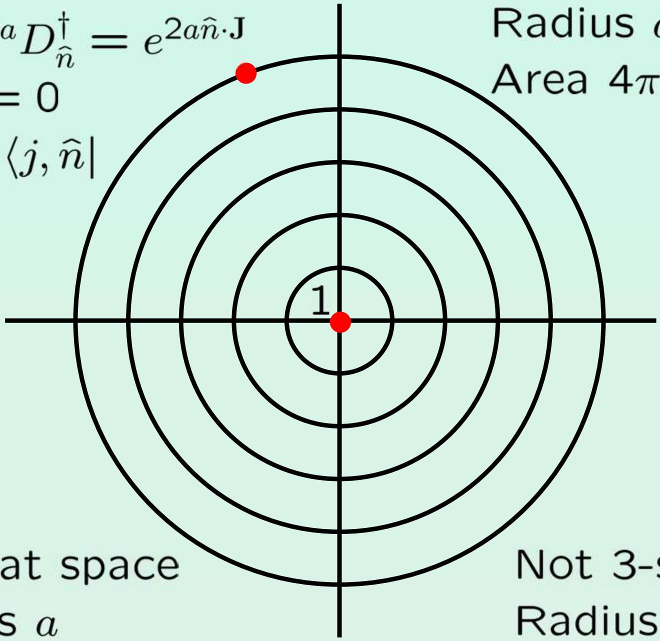
The angular diffusion of the POVM displacement is overwhelmed by the area exponentiation.

Nearly ballistic “collapse” to the coherent-state boundary at *a* = infinity.

$E = D_{\hat{n}} e^{J_z 2a} D_{\hat{n}}^{\dagger} = e^{2a\hat{n}\cdot\mathbf{J}}$

$E = 1$ at $a = 0$

$E \xrightarrow[a\to\infty]{} e^{2aj}|j,\hat{n}\rangle\langle j,\hat{n}|$

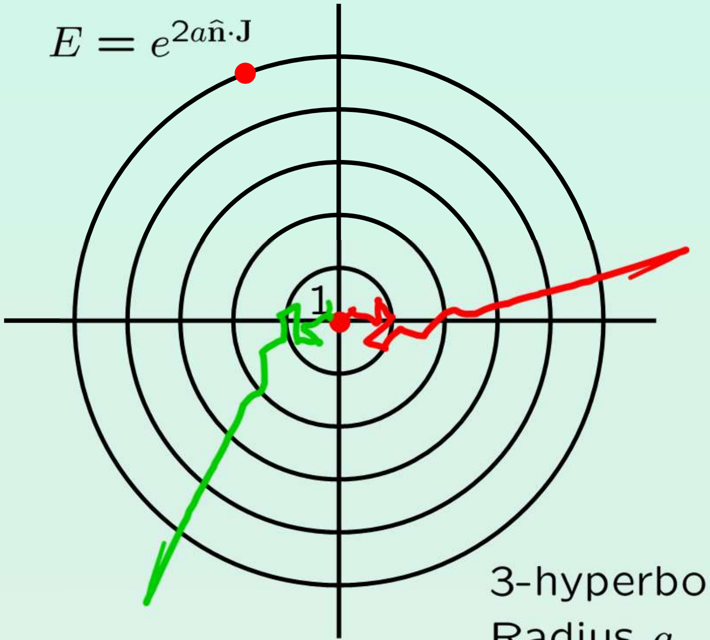


3-hyperboloid
Radius a
Area $4\pi\,\sinh^2 a$

Not flat space
Radius a
Area $4\pi a^2$

Not 3-sphere
Radius a
Area $4\pi\,\sin^2 a$

$E = e^{2a\hat{\mathbf{n}}\cdot\mathbf{J}}$



3-hyperboloid
Radius a
Area $4\pi\,\sinh^2 a$

Principal vs. chaotic instruments

SPQM and ISM are principal universal instruments, for which the universal instrumental group is finite-dimensional, because the nonlinear quadratic term has special properties. Principal instruments are very special: they approach a strong measurement of coherent states asymptotically; these instruments are thus universal (pre-quantum) and structure any Hilbert space in which they are represented. *They are what Heisenberg and Einstein meant by identifying what is observable.*

Generic measurements—e.g., two components of spin—have an infinite-dimensional universal instrumental group because of the nonlinear quadratic term. They do not have a representation-independent strong measurement, and the evolution of the instrument devolves into

chaos

Chaotic instruments

chaos

A universal chaotic instrument evolves stochastically into an increasing number of Lie-group dimensions. These Lie-group dimensions correspond to higher and higher powers of the measured observables and thus to finer and finer scales on a classical phase phase (sensitivity to initial conditions). Quantum chaos is what happens when the Lie-group dimensions, the higher powers, and the finer scales are cut off in a finite-dimensional Hilbert-space representation. All this is might be quantified by the entropies of the Kraus-operator distribution functions.

Discovery of universal chaotic instruments—and their quantum counterparts in finite-dimensional representations—promises a new group-theoretic method for analyzing quantum chaos and dynamical complexity.

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