Quantum metrology: dynamics vs. entanglement

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II. Quantum information perspective
III. Beyond the Heisenberg limit
IV. Two-component BECs

Appendix. Quantum metrology and resources

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Quantum circuits in this presentation were set using the LaTeX package Qcircuit, developed at the University of New Mexico by Bryan Eastin and Steve Flammia. The package is available at http://info.phys.unm.edu/Qcircuit/.
I. Ramsey interferometry and cat states

Herod’s Gate/King David’s Peak
Walls of Jerusalem NP
Tasmania
Ramsey interferometry

$|\uparrow\rangle$  \hspace{2cm} $|\downarrow\rangle$  \hspace{2cm} $|\frac{\uparrow}{\sqrt{N}}\rangle$  \hspace{2cm} $|\frac{\downarrow}{\sqrt{N}}\rangle$

$H = \frac{1}{2} \hbar \omega \sigma_z + \frac{1}{\sqrt{N}} e^{i \omega T} |\uparrow\rangle \langle \uparrow| - \frac{1}{\sqrt{N}} e^{i \omega T} |\downarrow\rangle \langle \downarrow|$

$R_\uparrow(\omega T) |\uparrow\rangle = \frac{1}{2} (1 - \cos \omega T) |\uparrow\rangle$

$R_\downarrow(\omega T) |\downarrow\rangle = \frac{1}{2} (1 + \cos \omega T) |\downarrow\rangle$

$\langle \sigma_z \rangle = - \cos \omega T$

$\Delta \sigma_z = \sqrt{1 - \cos^2 \omega T} = | \sin \omega T |$

$\Delta(\omega T) = \frac{1}{\sqrt{N}} |d(\text{signal})/d(\omega T)|$

$N$ independent “atoms”

Shot-noise limit

Frequency measurement

Time measurement

Clock synchronization
Cat-state Ramsey interferometry


\[ R_\lambda(\pi/2) \text{ on first atom} \]

\[ \text{C-NOT on rest} \]

\[ R_\theta(\pi/2) \text{ on first atom} \]

\[ \langle \sigma_z \rangle = -\cos N\omega T \]

\[ \Delta \sigma_z = \sqrt{1 - \cos^2 N\omega T} = |\sin N\omega T| \]

\[ \Delta(\omega T) = \frac{1}{\sqrt{\nu}} \left| \frac{d(\text{signal})}{d(\omega T)} \right| \]

\[ \nu = \text{(number of trials)} \]

N cat-state atoms

Fringe pattern with period \(2\pi/N\)

It's the entanglement, stupid.
II. Quantum information perspective

Cable Beach
Western Australia
Quantum information version of interferometry

Shot-noise limit

$U_\phi = e^{-iZ\phi/2}$

\[
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)
\]

\[
\frac{1}{\sqrt{2}}(e^{-i\phi/2}|0\rangle + e^{i\phi/2}|1\rangle)
\]

\[
\cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle)
\]

Fringe pattern with period $2\pi/N$

$[\cos(N\phi/2)|0\rangle - i\sin(N\phi/2)|1\rangle]|00\rangle$

$N = 3$

Cat state

$|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|11\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Heisenberg limit

Quantum states and operators for interferometry.
Cat-state interferometer

State preparation

Measurement

$U = e^{-i\hbar \phi}$

$h = \sum_{j=1}^{N} h_j$

Single-parameter estimation

ancilla

$U_\phi = e^{-i\hbar_1 \phi}$
$U_\phi = e^{-i\hbar_2 \phi}$
$U_\phi = e^{-i\hbar_3 \phi}$
Heisenberg limit


\[ U = e^{-i h \phi}, \quad h = \sum_{j=1}^{N} h_j \]

\[ \Delta \phi \geq \frac{1}{2 \Delta h} \geq \frac{1}{N (\Lambda - \lambda)} \]

\[ \Delta h \leq \frac{1}{2} \| h \| = \frac{1}{2} N (\Lambda - \lambda) \]

Generalized uncertainty principle (Cramér-Rao bound)

Separable inputs

\[ \Delta h \leq \frac{1}{2} \sqrt{N (\Lambda - \lambda)} \]

\[ \Delta \phi \geq \frac{1}{\sqrt{N (\Lambda - \lambda)}} \]

W, \ U_\phi, \ and \ ancilla \ measurements \ can \ be \ interleaved.
Achieving the Heisenberg limit

\[ |S\rangle \quad U_\phi = e^{-i\hbar_1 \phi} \quad M \]

\[ |S\rangle \quad V \quad U_\phi = e^{-i\hbar_2 \phi} \quad M \]

\[ |S\rangle \quad U_\phi = e^{-i\hbar_3 \phi} \quad M \]

**Cat state**
\[
\frac{1}{\sqrt{2}} (|\Lambda, \ldots, \Lambda\rangle + |\lambda, \ldots, \lambda\rangle)
\]

\[
\frac{1}{\sqrt{2}} (e^{-iN\Lambda \phi}|\Lambda, \ldots, \Lambda\rangle + e^{-iN\lambda \phi}|\lambda, \ldots, \lambda\rangle)
\]

\[
e^{-iN(\Lambda+\lambda)\phi/2} \left( \cos[N(\Lambda - \lambda)\phi/2]|\Lambda, \ldots, \Lambda\rangle - i \sin[N(\Lambda - \lambda)\phi/2]|\lambda, \ldots, \lambda\rangle \right)
\]

**Fringe pattern with period**
\[
\Delta \phi = \frac{1}{2\pi/N(\Lambda - \lambda)}
\]
Is it entanglement? It’s the entanglement, stupid.

But what about?

- Flip half the spins in a cat state, and you get a state with the same amount of entanglement, but one that is worthless for metrology.

- There are states with far more bipartite entanglement than the cat state—up to about $N/2$ e-bits for equal bipartite splits—yet they are useless for metrology.

- Measurement sensitivity and optimal initial state depend on local Hamiltonians $h_j$, but entanglement measures are usually constructed to be independent of such mundane details.

We need a generalized notion of entanglement that includes information about the physical situation, particularly the relevant Hamiltonian.
III. Beyond the Heisenberg limit

Echidna Gorge
Bungle Bungle Range
Western Australia
Beyond the Heisenberg limit

The purpose of theorems in physics is to lay out the assumptions clearly so one can discover which assumptions have to be violated.

\[ |S\rangle \quad V \quad U_\phi = e^{-i\hbar \phi} \quad W \]

ancilla

Cat state does the job.

Nonlinear Ramsey interferometry

\[ \Delta \phi \geq \frac{1}{2 \Delta \hbar} \geq \frac{1}{||h||} = \frac{1}{\mathcal{N}^k (\land^k - \lambda^k)} \]

Metrologically relevant $k$-body coupling

\[ h = \left( \sum_{j=1}^{N} h_j \right)^k = \sum_{j_1, \ldots, j_k} h_{j_1} h_{j_2} \cdots h_{j_k} \]

$N^k$ terms in sum

\[ ||h|| = \mathcal{N}^k (\land^k - \lambda^k) \]
Improving the scaling with $N$ without entanglement


$$|S\rangle$$

$$|S\rangle$$

$$|S\rangle$$

$$U_\phi = e^{-i\hbar \phi}$$

$$M$$

$$M$$

$$M$$

Product input

Product measurement

$$h = \left( \sum_{j=1}^{N} Z_j / 2 \right)^k = J_z^k$$

$$\Delta \phi \sim \frac{1}{N^{k-1/2}}$$
Improving the scaling with $N$ without entanglement.

Two-body couplings

\[ \chi_1 n_1 + \chi_2 n_2 = \frac{1}{2}(\chi_1 + \chi_2)N + (\chi_1 - \chi_2)J_z \]

\[ N = n_1 + n_2, \quad J_z = \frac{1}{2}(n_1 - n_2) \]

\[ \Delta \phi = \frac{1}{\sqrt{N}} \]

\[ \chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_1 \chi_2 n_1 n_2 = \frac{1}{4}(\chi_1 + \chi_2 + 2\chi_{12})N^2 + (\chi_1 - \chi_2)NJ_z + (\chi_1 + \chi_2 - 2\chi_{12})J_z^2 \]

\[ \Delta \phi = \frac{1}{N^{3/2}} \]

Improving the scaling with \( N \) without entanglement.

Two-body couplings

\[
\chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2 = \frac{1}{4}(\chi_1 + \chi_2 + 2\chi_{12}) N^2 + (\chi_1 - \chi_2) NJ_z \equiv \phi + (\chi_1 + \chi_2 - 2\chi_{12}) J_z^2 = 0
\]

\[
\Delta \phi = \frac{1}{N^{3/2}}
\]

Super-Heisenberg scaling from nonlinear dynamics, without any particle entanglement

Scaling robust against decoherence

IV. Two-component BECs

Pecos Wilderness
Sangre de Cristo Range
Northern New Mexico
Two-component BECs

Nonlinear BEC Ramsey interferometer

$^{87}$Rb atoms cooled to spatial ground state in hyperfine level $|F = 1; M_F = -1\rangle$. Other relevant hyperfine level is $|F = 2; M_F = 1\rangle$, which sees the same trapping potential.

- $\pi/2$ transition.
- Atoms in $|1\rangle$ see nonlinear phase shift $\frac{1}{2}(g_{11}n_1^2 + g_{12}n_1n_2)$, and atoms in $|2\rangle$ see nonlinear phase shift $\frac{1}{2}(g_{12}n_1n_2 + g_{22}n_2^2)$, where $g_{j,k} = 4\pi\hbar^2 a_{j,k}/m$.
- $\pi/2$ transition.
- Measure number of atoms in $|1\rangle$ and $|2\rangle$.

$$a_{11} = 100.40a_0 , \quad a_{22} = 95.00a_0 , \quad a_{12} = 97.66a_0$$

$$\frac{1}{2}(a_{11} - a_{22}) = 2.70a_0 , \quad \frac{1}{2}(a_{11} + a_{22}) - a_{12} = 0.04a_0$$

Nearly pure $NJ_z$ coupling to measure $\gamma = \frac{1}{2}(g_{11} - g_{22})$
Two-component BECs

Isotropic, harmonic trap with bare ground-state width $R_0$

Different spatial wave functions

$$\left( \begin{array}{c} \text{critical atom number} \\ \text{number} \end{array} \right) = N_c = \frac{R_0}{a}$$

$$\frac{\tau_{\text{phase}}}{\tau_{\text{spatial}}} \sim \frac{1}{10}$$

$\tau_{\text{spatial}}$ is time scale for divergence of the spatial wave functions for the two hyperfine levels.

J. E. Williams, PhD dissertation, University of Colorado, 1999.
Two-component BECs

Isotropic, harmonic trap with bare ground-state width $R_0$

**Different spatial wave functions**

\[
\frac{R}{R_0} \sim \left( \frac{N}{N_c} \right)^{1/5}
\]

\[
\frac{g}{R^3} \sim \frac{g}{R_0^3} \left( \frac{N_c}{N} \right)^{3/5}
\]

\[
\Delta \gamma \sim \frac{1}{N^{9/10}}
\]

$\tau_{\text{phase}} \approx \frac{1}{10}$

$\tau_{\text{spatial}}$ is time scale for divergence of the spatial wave functions for the two hyperfine levels.

**Renormalization of scattering strengths**

Let’s start over.
Two-component BECs

Anisotropic, nonharmonic trap: $d$ dimensions loosely confined by a power-law potential $V = \frac{1}{2}kr^q$, with bare ground-state width $R_0 \approx (\hbar^2/mk)^{1/(q+2)}$; $D = 3 - d$ dimensions tightly confined in a harmonic potential with bare ground-state width $s$.

Different spatial wave functions

$$N_c = \frac{R_0}{a} \left( \frac{s}{R_0} \right)^D$$

$\tau_{\text{spatial}}$ is time scale for divergence of the spatial wave functions for the two hyperfine levels.

$$\frac{\tau_{\text{phase}}}{\tau_{\text{spatial}}} \sim \frac{1}{2} \frac{q}{2q + d} \sqrt{\frac{d}{3q + d}}$$

Renormalization of scattering strengths

$$\frac{R}{R_0} \sim \left( \frac{N}{N_c} \right)^{1/(q+d)}$$

$$\frac{g}{s^D R^d} \sim \frac{g}{s^D R_0^d} \left( \frac{N_c}{N} \right)^{d/(q+d)}$$

$$\Delta \gamma \sim \frac{1}{N(3q+d)/2(q+d)}$$
Two-component BECs

Anisotropic, nonharmonic trap: \( d \) dimensions loosely confined by a power-law potential \( V = \frac{1}{2}kr^q \), with bare ground-state width \( R_0 \sim (\hbar^2/mk)^{1/(q+2)} \); \( D = 3 - d \) dimensions tightly confined in a harmonic potential with bare ground-state width \( s \).

Two-body elastic losses

\[
\frac{\tau_{\text{phase}}}{\tau_{\text{losses}}} \sim \frac{1}{26}
\]

Imprecise determination of \( N \)

Counting atoms to accuracy \( \Delta N/N \sim 1\% \) would be more than sufficient.

? Perhaps ?

With hard, low-dimensional trap
Appendix. Quantum metrology and resources
Making quantum limits relevant

Optimal sensitivity: \[ \Delta \omega \sim \frac{1}{TN} \]

The serial resource, \( T \), and the parallel resource, \( N \), are equivalent and interchangeable, \textit{mathematically}.

The serial resource, \( T \), and the parallel resource, \( N \), are not equivalent and not interchangeable, \textit{physically}.

Information science perspective
\textit{Platform independence}

Physics perspective
\textit{Distinctions between different physical systems}
Making quantum limits relevant. One metrology story

Resources

- Overall measurement time $\tau$ or inverse bandwidth (the "classical serial resource")
- Coherent interaction time $T$ for an individual probe (the "quantum serial resource")
- Rate $R$ at which systems can be deployed [$R(\tau - T) = n$ is the "classical parallel resource"]
- Entanglement within each probe consisting of $N$ systems ($N$ is the "quantum parallel resource")

Problem

Given $\tau$ and $R$ and a decoherence rate $\Gamma$, what is the best strategy for estimating frequency $\omega = \phi/T$?

The answer is well known for squeezed-state optical interferometry and has recently been analyzed in detail for linear Ramsey interferometry: The quantum resources—extended coherent evolution and entanglement—are useful only if $\Gamma \tau \lesssim 1$ and $R\tau \gg 1$.

One metrology story

<table>
<thead>
<tr>
<th>Range of $\Gamma \tau$</th>
<th>$T$</th>
<th>$N$</th>
<th>$\nu$</th>
<th>$n$</th>
<th>$\Delta \omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\nu_{\min} \Gamma}{R} \leq \Gamma \tau &lt; \frac{2\nu_{\min} \Gamma}{R}$</td>
<td>$T_s$</td>
<td>1</td>
<td>$\nu_{\min}$</td>
<td>$\nu_{\min}$</td>
<td>$\frac{e^{\Gamma T_s}}{T_s \sqrt{\nu_{\min}}}$</td>
</tr>
<tr>
<td>$\frac{2\nu_{\min} \Gamma}{R} \leq \Gamma \tau &lt; \sqrt{\frac{2\nu_{\min} \Gamma}{R}}$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{R \tau}{2 \nu_{\min}}$</td>
<td>$\nu_{\min}$</td>
<td>$\frac{R \tau}{2}$</td>
<td>$\frac{4 \sqrt{\nu_{\min}}}{R \tau^2} e^{\Gamma R \tau^2/4 \nu_{\min}}$</td>
</tr>
<tr>
<td>$\sqrt{\frac{2\nu_{\min} \Gamma}{R}} \leq \Gamma \tau &lt; 1$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{1}{\Gamma \tau}$</td>
<td>$\frac{\Gamma R \tau^2}{2}$</td>
<td>$\frac{R \tau}{2}$</td>
<td>$\frac{2 \sqrt{2e}}{\tau \sqrt{R / \Gamma}}$</td>
</tr>
<tr>
<td>$\Gamma \tau \geq 1$</td>
<td>$T_p$</td>
<td>1</td>
<td>$R(\tau - T_p)$</td>
<td>$R(\tau - T_p)$</td>
<td>$\frac{e^{\Gamma T_p}}{T_p \sqrt{R(\tau - T_p)}}$</td>
</tr>
</tbody>
</table>

$T_s = \tau - \nu_{\min}/R$

$T_p = \frac{3/2 + \Gamma \tau - \sqrt{(3/2 + \Gamma \tau)^2 - 4 \Gamma \tau}}{2 \Gamma} \rightarrow 1/\Gamma$ when $\Gamma \tau \gg 1$
One metrology story

<table>
<thead>
<tr>
<th>Qubit starvation</th>
<th>Range of $R\tau/\nu_{\text{min}}$</th>
<th>$T$</th>
<th>$N$</th>
<th>$\nu$</th>
<th>$n$</th>
<th>$\Delta\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 \leq \frac{R\tau}{\nu_{\text{min}}} &lt; 2$</td>
<td>$\sim \tau$</td>
<td>$1$</td>
<td>$\nu_{\text{min}}$</td>
<td>$\nu_{\text{min}}$</td>
<td>$\sim \frac{1}{\tau}$</td>
</tr>
<tr>
<td>Cat-state regime</td>
<td>$2 \leq \frac{R\tau}{\nu_{\text{min}}} &lt; \sqrt{\frac{2R}{\nu_{\text{min}}\Gamma}}$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{R\tau}{2\nu_{\text{min}}}$</td>
<td>$\nu_{\text{min}}$</td>
<td>$\frac{R\tau}{2}$</td>
<td>$\sim \frac{1}{\tau^2}$</td>
</tr>
<tr>
<td>Cat-state transition</td>
<td>$\sqrt{\frac{2R}{\nu_{\text{min}}\Gamma}} \leq \frac{R\tau}{\nu_{\text{min}}} &lt; \frac{R}{\nu_{\text{min}}\Gamma}$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{1}{\Gamma\tau}$</td>
<td>$\frac{\Gamma R\tau^2}{2}$</td>
<td>$\frac{R\tau}{2}$</td>
<td>$\sim \frac{1}{\tau}$</td>
</tr>
<tr>
<td>Decoherence dominance</td>
<td>$\frac{R\tau}{\nu_{\text{min}}} \geq \frac{R}{\nu_{\text{min}}\Gamma}$</td>
<td>$\sim \frac{1}{\Gamma}$</td>
<td>$1$</td>
<td>$\sim R\tau$</td>
<td>$\sim R\tau$</td>
<td>$\sim \frac{1}{\sqrt{\tau}}$</td>
</tr>
</tbody>
</table>