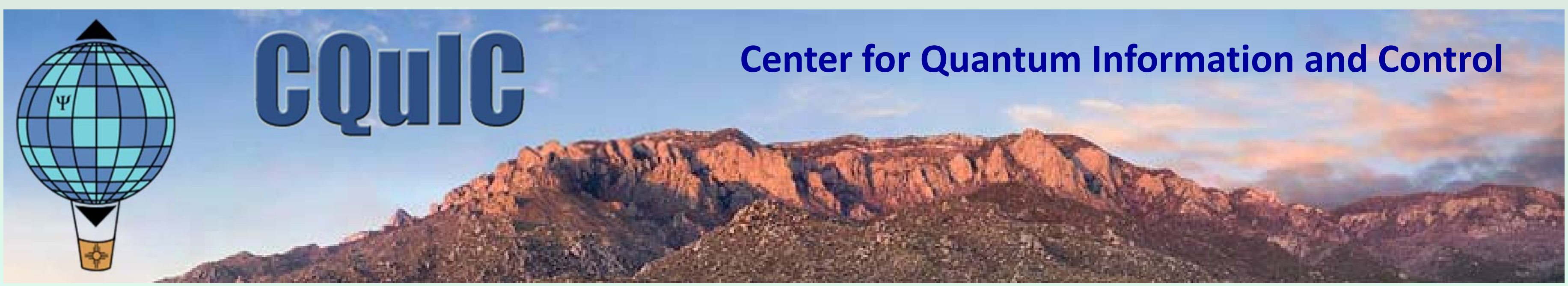
Quantum Gerrymandering: Positivity, Bias, and Anisotropy Among Quantum States

I. Introduction: Geometry, Tomography, and Metrology
II. How gerrymandered is quantum state space?
III. Tomography and the spherical cow
IV. Anisotropy in qutrit tomography

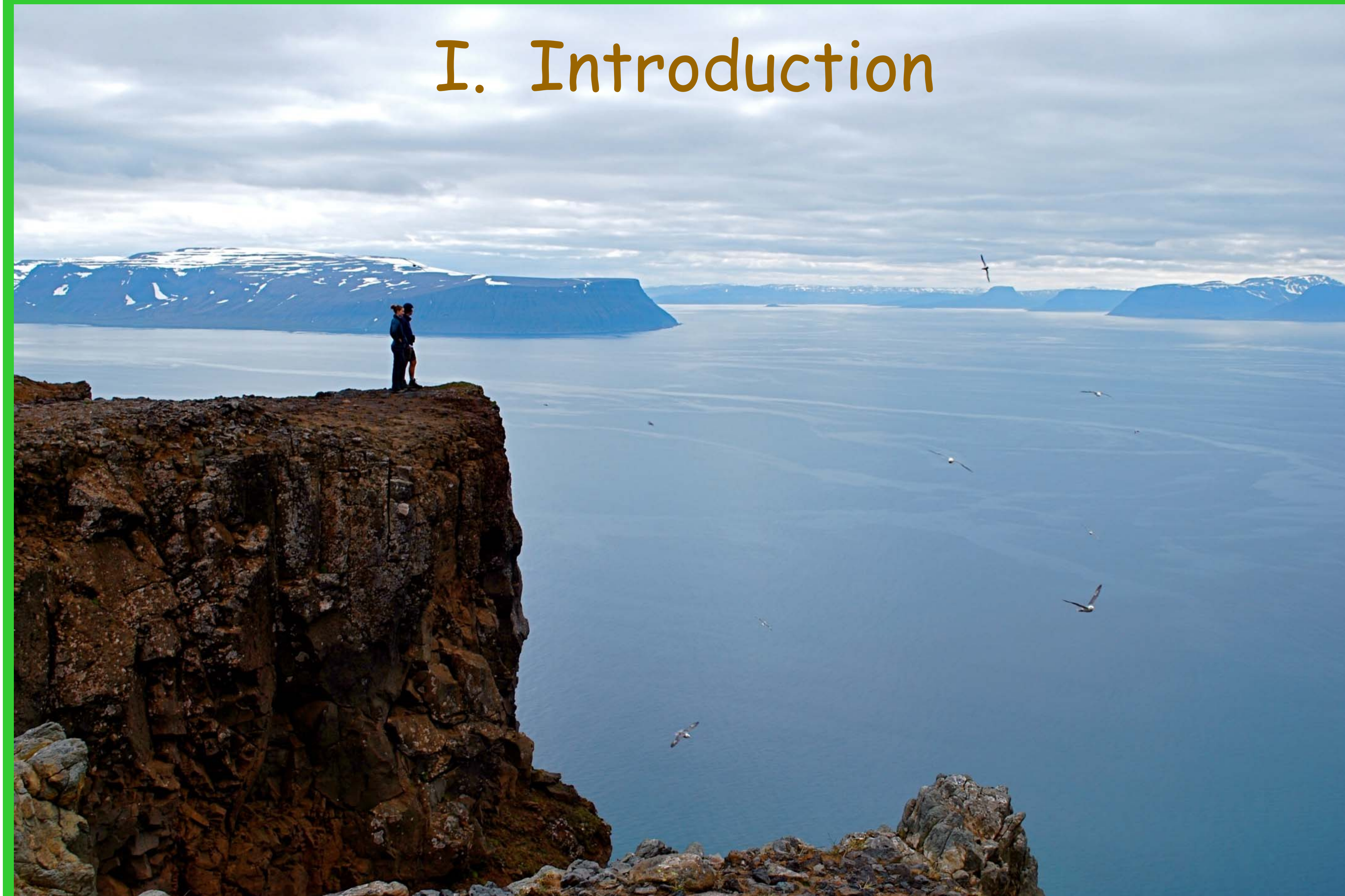
Jonathan A. Gross and Carlton M. Caves

Center for Quantum Information and Control, University of New Mexico

https://cquic.unm.edu/



I. Introduction



Holstrandir Peninsula overlooking Ísafjarðardjúp
Westfjords, Iceland

State space

Classical

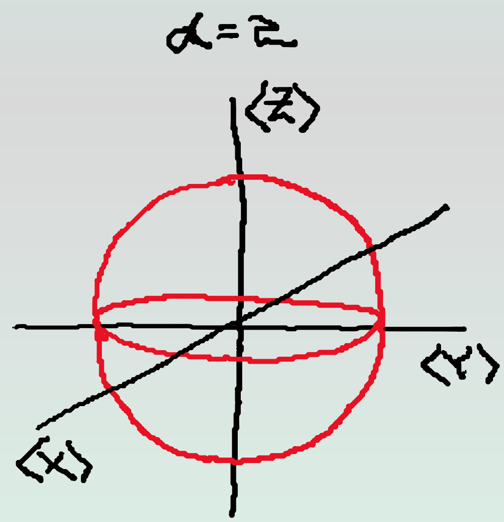
$$\left\{ \mathbf{p} = (p_1, \ldots, p_n) \;\middle|\; p_j \geq 0,\ \sum_j p_j = 1 \right\}$$

$(n-1)$-dimensional *probability simplex*,
regular polyhedron with n vertices,
is a convex set
with extreme points (certainty):
purity $\sum_j p_j^2 = 1$

Quantum

$$\left\{ d \times d \text{ matrix } \rho \;\middle|\; \rho \geq 0,\ \mathrm{tr}(\rho) = 1 \right\}$$

(d^2-1)-dimensional *Bloch body*
is a convex set
with extreme points (pure states):
purity $\mathrm{tr}(\rho^2) = 1$



Bloch ball

$$\rho = \frac{1}{2}(I + \mathbf{S} \cdot \boldsymbol{\sigma})$$

$$\langle \boldsymbol{\sigma} \rangle = \mathrm{tr}(\rho \boldsymbol{\sigma}) = \mathbf{S}$$

State space

Classical

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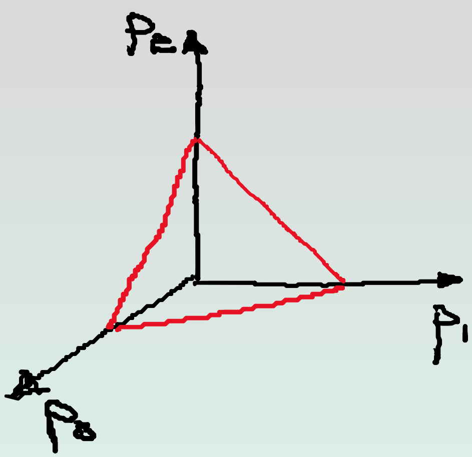
$n=3$



Quantum

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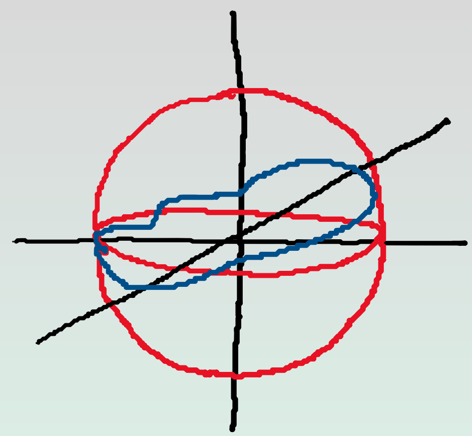

8-dimensional Bloch body

4-dimensional manifold
of pure states
on surface of ball

State space

Classical

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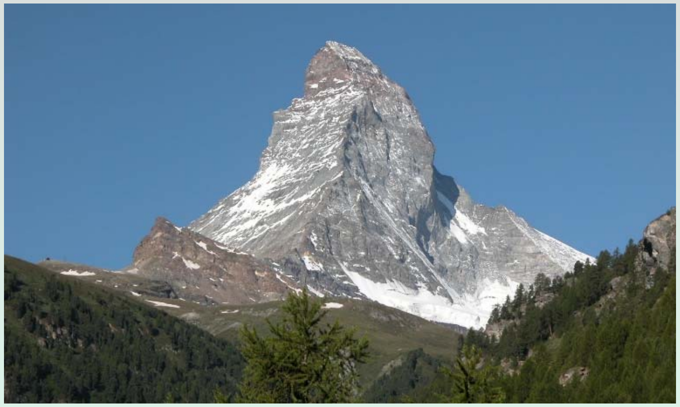
(d^2-1)-dimensional *Bloch body*
is a convex set
with extreme points (pure states):
purity $\mathrm{tr}(\rho^2) = 1$.

Pure states are a
$(2d-2)$-dimensional manifold
pasted onto the surface
of the (d^2-1)-dimensional ball
of unit radius.

Near an extreme point



Matterhorn
Swiss Alps

Mt. Sir Alexander
Canadian Rockies

State space

Classical

$$\left\{\mathbf{p} = (p_1, \ldots, p_n) \;\middle|\; p_j \geq 0,\ \sum_j p_j = 1\right\}$$

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Near an extreme point

Gerrymandering

Exploring the corners of state space

Know the nooks and crannies of your neighborhood.

Geometry

Tomography

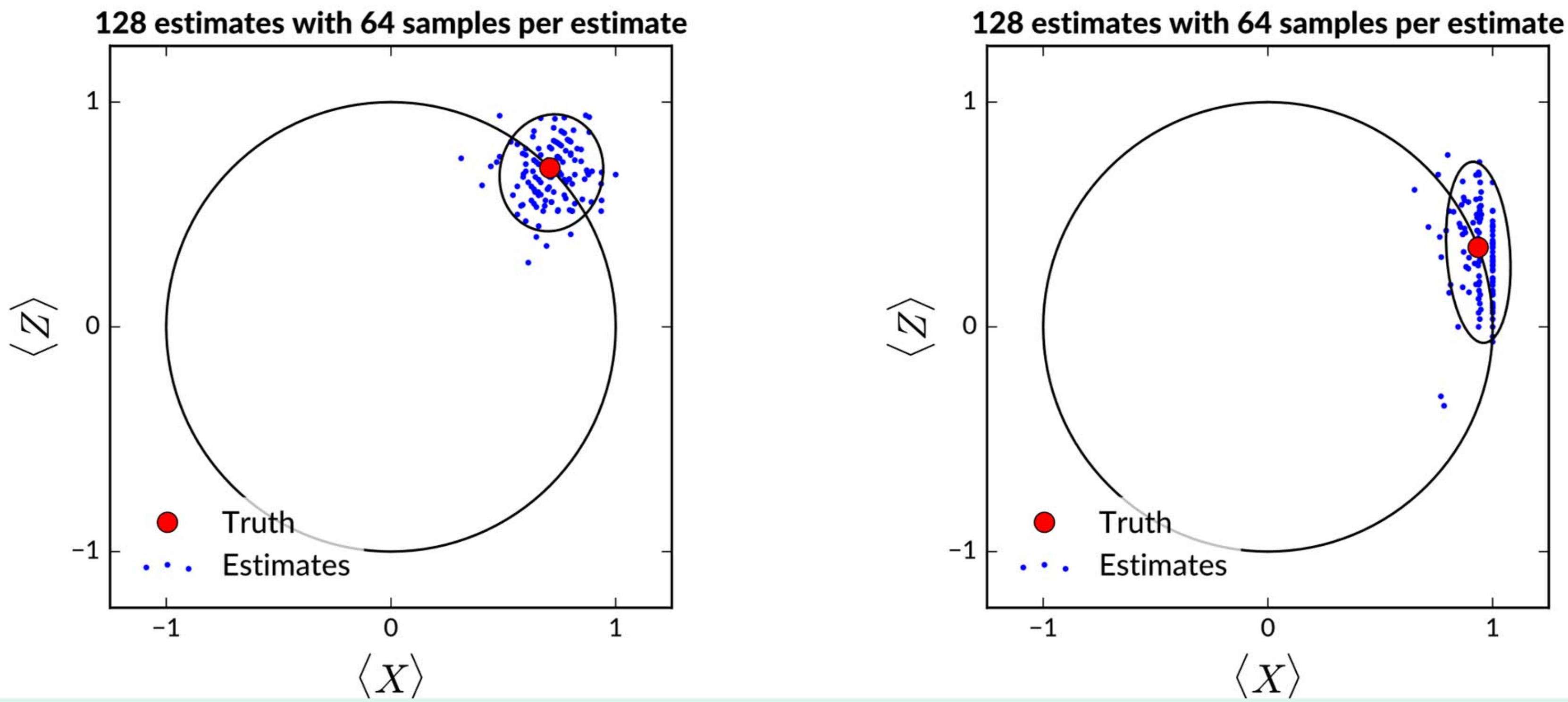
Metrology

Low-rank tomography is all about working the corners of state space.

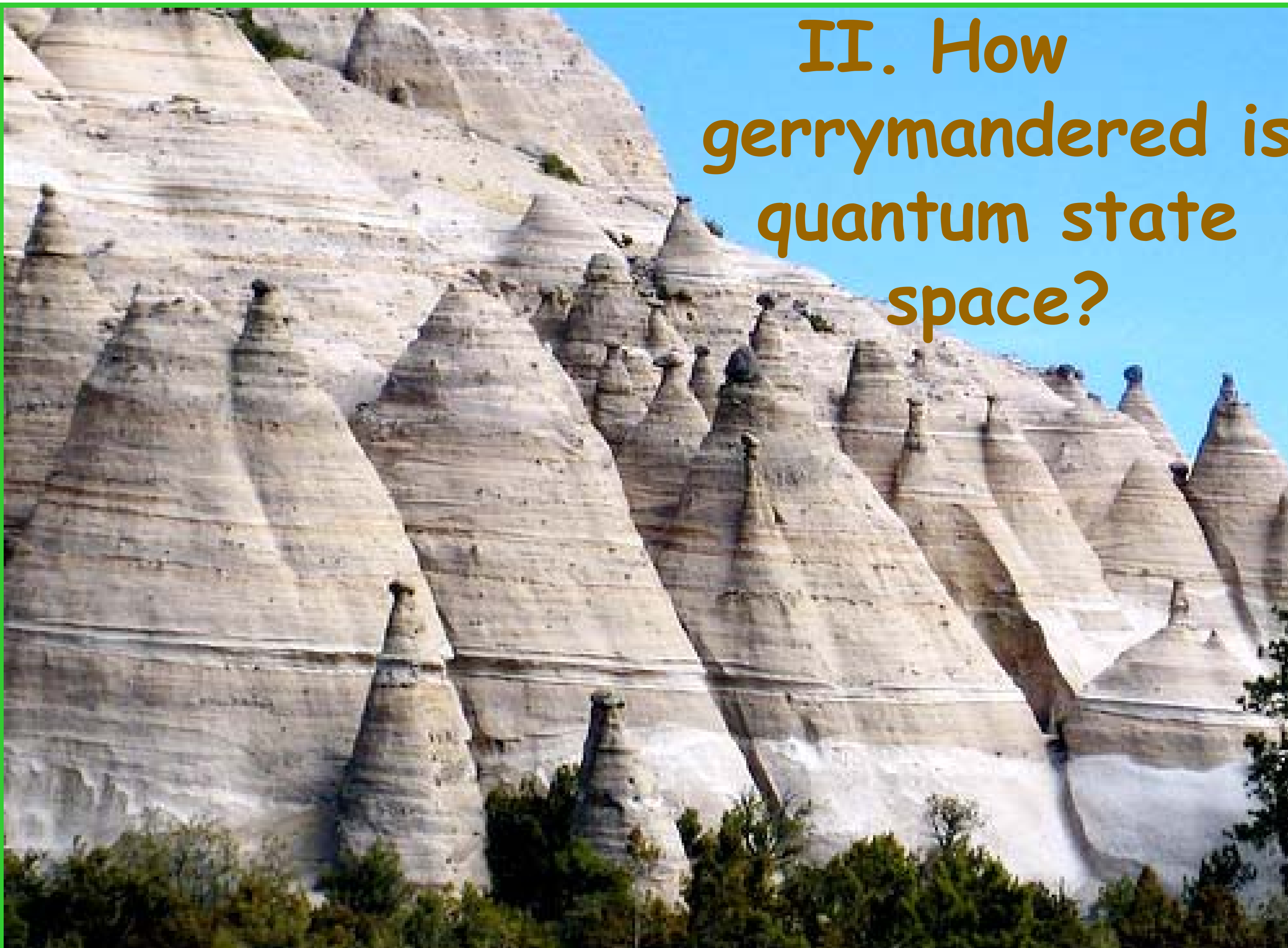
Tomography is the ultimate metrology. Fisher information is the tool.

Geometry, tomography, and metrology

Qubit
Measure *X* and *Z*
Maximum-likelihood estimation
Classical Fisher information



II. How gerrymandered is quantum state space?



Tent Rocks
Kasha-Katuwe National Monument
Northern New Mexico

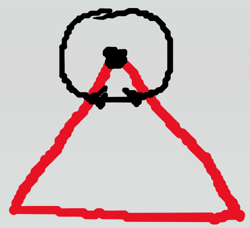
Fractional solid angle of physicality

Classical

$n=2$

$$\Omega_2 = \frac{1}{2}$$

$n=3$

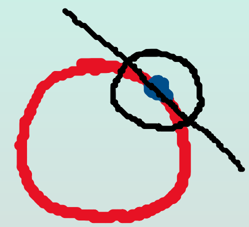


$$\Omega_3 = \frac{1}{6}$$

$$\Omega_n = \frac{\displaystyle\int_{S\cap\Delta} d\Delta_{n-1}}{\displaystyle\int_S d\Delta_{n-1}}$$

Quantum

$d=2$



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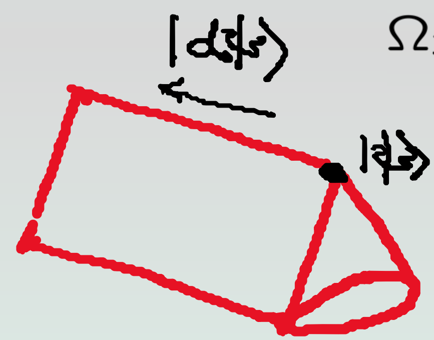
$d=3$



$$\Omega_3 = \frac{1}{6} - \frac{\sqrt{3}}{4\pi} \simeq \frac{1}{35}$$

$$\Omega_d = \frac{\displaystyle\int_{S\cap\Delta} d\Delta_{d-1} \prod_{2\leq k<l} (\lambda_k - \lambda_l)^2}{\displaystyle\int_S d\Delta_{d-1}}$$

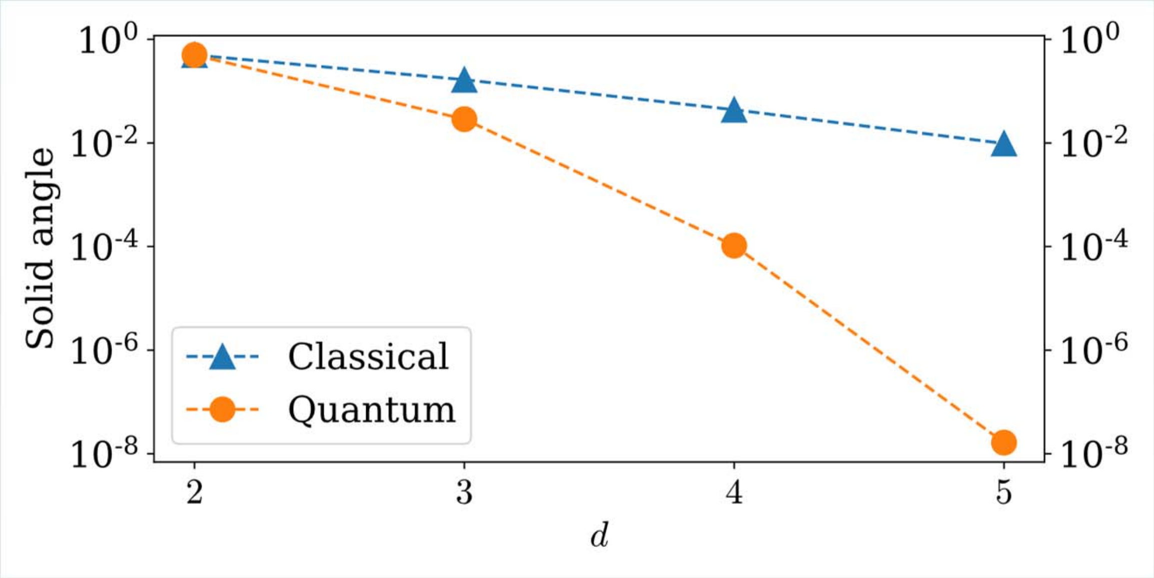
Fractional solid angle of physicality

Classical

$$\Omega_n = \frac{\int_{S\cap\Delta} d\Delta_{n-1}}{\int_S d\Delta_{n-1}}$$

Quantum

$$\Omega_d = \frac{\int_{S\cap\Delta} d\Delta_{d-1} \prod_{2\leq k<l} (\lambda_k - \lambda_l)^2}{\int_S d\Delta_{d-1}}$$



$$\Omega_n \sim \underbrace{\frac{1}{e}\sqrt{\frac{n}{2}}\left(\frac{e}{2\pi c^2(n-1)}\right)^{(n-1)/2}}_{\sim 1/\sqrt{\Gamma(n)}} \left[1 + \frac{31}{12}\frac{1}{n-1} + o\left(\frac{1}{n-1}\right)^2\right]$$

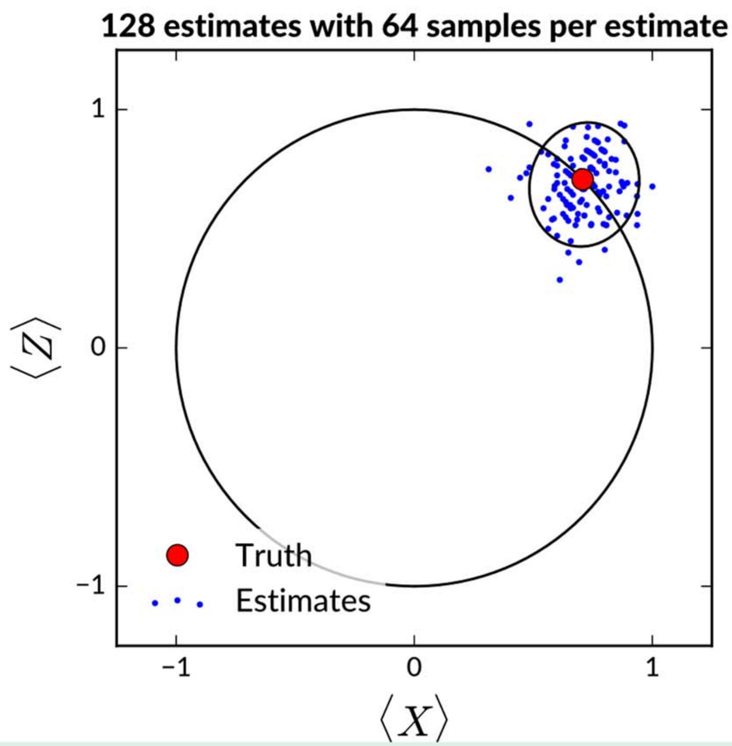
$$\Omega_d \underset{?}{\sim} \frac{1}{\sqrt{\Gamma(d^2)}}$$

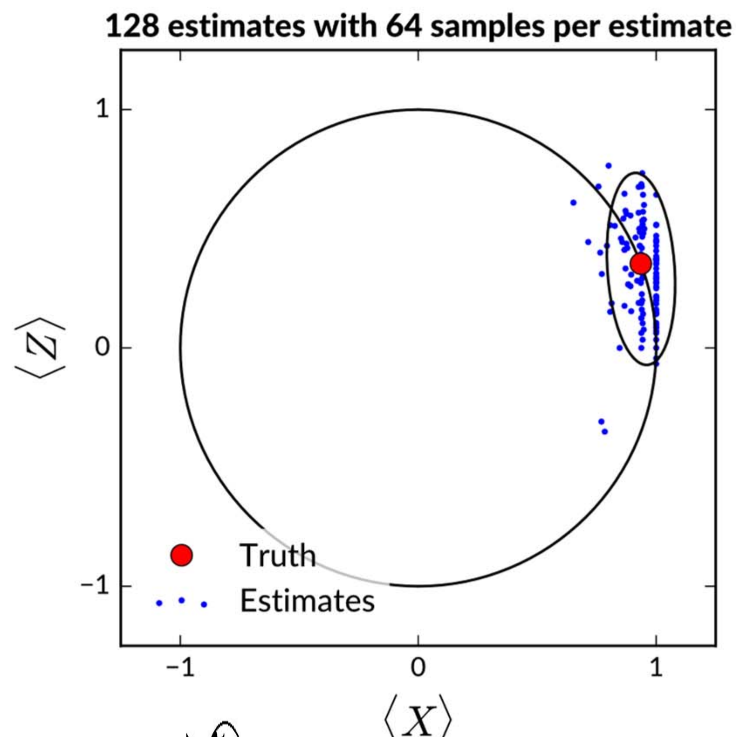
III. Tomography and the spherical cow

View from Cape Hauy
Tasman Peninsula, Tasmania

Tomography: Flat metric vs. Fisher metric

Qubit: Measure X and Z
Maximum-likelihood estimation
Classical Fisher information

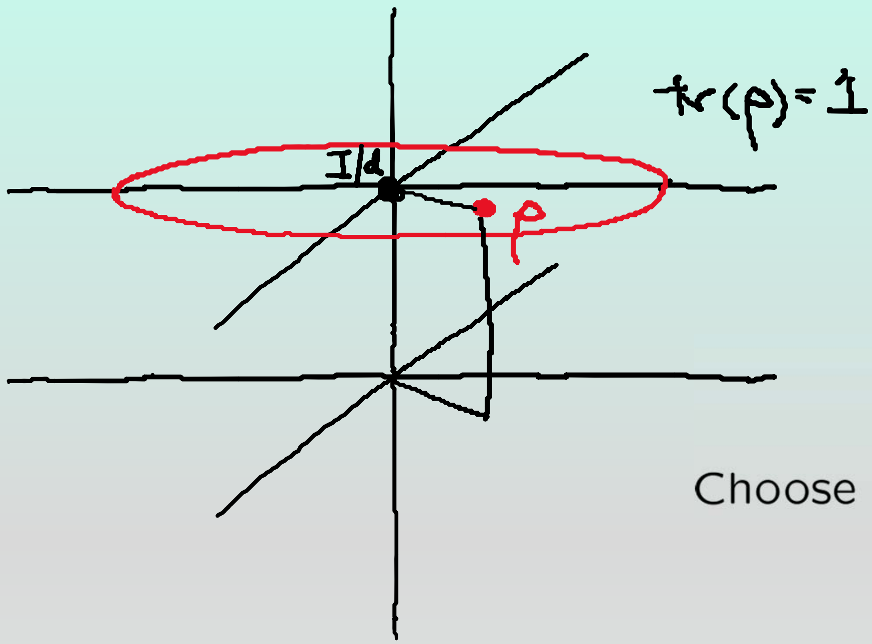




Hilbert-Schmidt (HS) flat metric: isotropic

Fisher metric: can it be isotropic?

Classical Fisher matrix



$$\rho = \frac{I}{d} + x^j X_j = \rho^\dagger$$

$$\mathrm{tr}(\rho) = 1$$

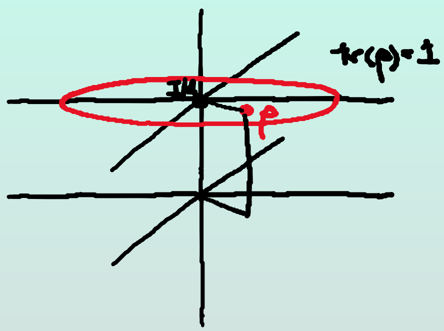
X_j are Hermitian: $X_j = X_j^\dagger$
X_j are traceless: $\mathrm{tr}(X_j) = 1$
Choose X_j to be HS orthogonal: $\mathrm{tr}(X_j X_k) = \delta_{jk}$

Make a measurement of a POVM $\{E^\xi \geq 0\}$;
completeness: $\sum_\xi E^\xi = I$.
Probability for result ξ is $p(\xi|\rho) = \mathrm{tr}(\rho E^\xi)$.

Classical Fisher-information matrix

$$F_{jk} = \sum_\xi \frac{1}{p(\xi|\rho)} \frac{\partial p(\xi|\rho)}{\partial x^j} \frac{\partial p(\xi|\rho)}{\partial x^k} = \sum_\xi \frac{\mathrm{tr}(E^\xi X_j)\mathrm{tr}(E^\xi X_k)}{\mathrm{tr}(E^\xi \rho)}$$

Classical Fisher metric



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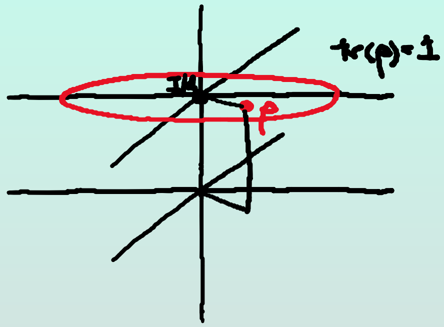
Classical Fisher-information metric

$$\mathcal{F}_\rho = \sum_\xi \frac{E^\xi \otimes E^\xi}{\mathrm{tr}(E^\xi \rho)} = \sum_\xi \frac{|E^\xi)(E^\xi|}{\mathrm{tr}(E^\xi \rho)}$$

HS inner product: $(A|B) = \mathrm{tr}(A^\dagger B)$

$$F_{jk} = \mathcal{F}_\rho(X_j, X_k) = (X_j|\mathcal{F}_\rho|X_k)$$

Classical Fisher metric



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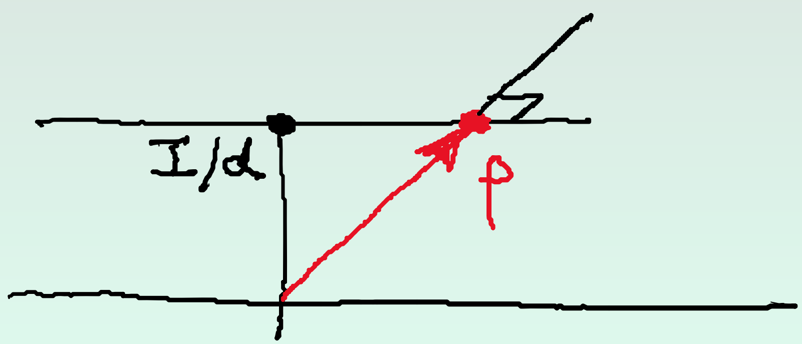
$$F_{jk} = \mathcal{F}_\rho(X_j, X_k) = (X_j|\mathcal{F}_\rho|X_k)$$

Extension (completeness) properties

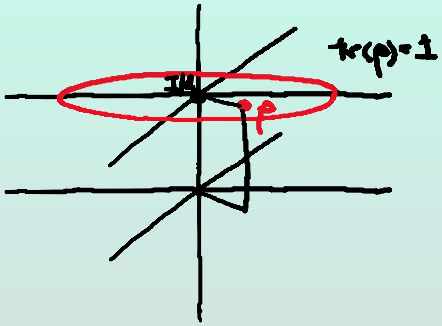
$$\mathcal{F}_\rho(\rho,\rho) = (\rho|\mathcal{F}_\rho|\rho) = \sum_\xi \mathrm{tr}(\rho E^\xi) = 1$$

$$\mathcal{F}_\rho(\rho, X_j) = (\rho|\mathcal{F}_\rho|X_j) = \sum_\xi \mathrm{tr}(E^\xi X_j) = 0$$

ρ is the unit vector orthogonal to the surface $\mathrm{tr}(\rho) = 1$



Classical Fisher metric



$$\mathcal{F}_\rho = \sum_\xi \frac{E^\xi \otimes E^\xi}{\mathrm{tr}(E^\xi \rho)} = \sum_\xi \frac{|E^\xi)(E^\xi|}{\mathrm{tr}(E^\xi \rho)}$$

$$F_{jk} = \mathcal{F}_\rho(X_j, X_k) = (X_j|\mathcal{F}_\rho|X_k)$$

Completeness: ρ is the unit vector orthogonal to the surface $\mathrm{tr}(\rho) = 1$, the traceless plane.

As a quantum operation, $\mathcal{F}_\rho$ is

positive: $\mathcal{F}_\rho(G) = \sum_\xi \frac{E^\xi G E^\xi}{\mathrm{tr}(\rho E^\xi)} \geq 0 \quad \text{if } G \geq 0;$

completely positive: $(A|\mathcal{F}_\rho|A) \geq 0$ for all operators A.

$(A|\mathcal{F}_\rho|A) > 0$ (metric/informational-completeness)

Special property

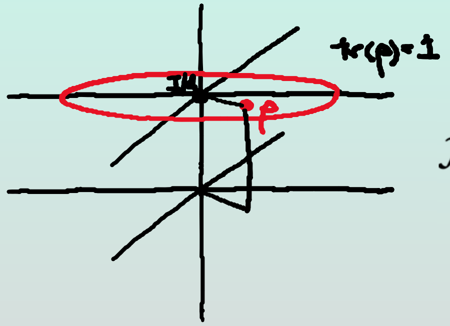
$$\mathcal{F}_\rho^\#(G) = \mathcal{F}_\rho|G) = \sum_\xi \frac{E^\xi\, \mathrm{tr}(E^\xi G)}{\mathrm{tr}(\rho E^\xi)} \geq 0 \quad \text{if } G \geq 0 \quad \Longleftrightarrow \quad \mathcal{F}_\rho^\# \text{ is positive}$$

If $\{E^\xi\}$ is a rank-one POVM,

$$\mathcal{F}_\rho^\#(A) = \mathcal{F}_\rho|A) = \mathcal{F}_\rho(A) \quad \Longleftrightarrow \quad \mathcal{F}_\rho = \mathcal{F}_\rho^\#$$

Are there spherical cows?

Can the Hilbert-Schmidt metric be the Fisher metric for a rank-one POVM?



$$\mathcal{F}_\rho = \sum_\xi \frac{E^\xi \otimes E^\xi}{\mathrm{tr}(E^\xi \rho)} = \sum_\xi \frac{|E^\xi)(E^\xi|}{\mathrm{tr}(E^\xi \rho)}$$

Completeness: ρ is the unit vector orthogonal to the surface $\mathrm{tr}(\rho) = 1$, the traceless plane.

Rank-one POVM: $\mathcal{F}_\rho = \mathcal{F}_\rho^\#$

The Hilbert-Schmidt metric is the flat, Cartesian metric:

$$\mathbf{I} = \sum_\alpha |\tau_\alpha)(\tau_\alpha| \quad \Longleftrightarrow \quad \mathbf{I}|A) = |A)$$

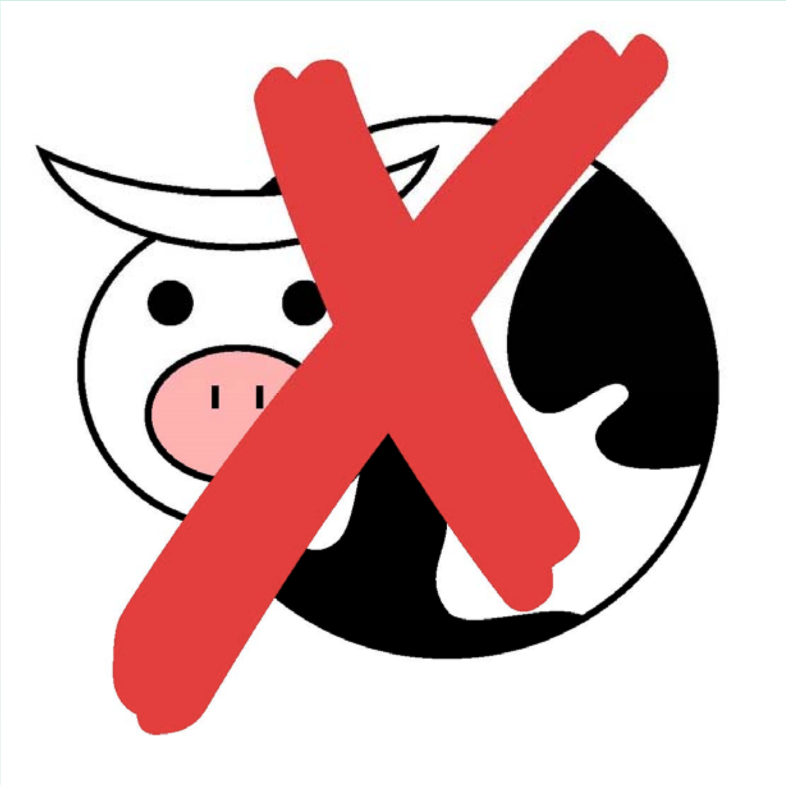
Project HS metric into traceless plane, allow rescaling, and make ρ the unit vector orthogonal to traceless plane:

$$\mathcal{G}_\rho = |I)(I| + s\big[\mathbf{I} - |I)(\rho|\big]\mathbf{I}\big[\mathbf{I} - |\rho)(I|\big]$$

The HS metric is a Fisher metric for a rank-one POVM—$\mathcal{G}_\rho$ can be made equal to $\mathcal{G}_\rho^\#$—if and only if ρ is the maximally mixed state ($\rho = I/d$) or $d = 2$ (a qubit).

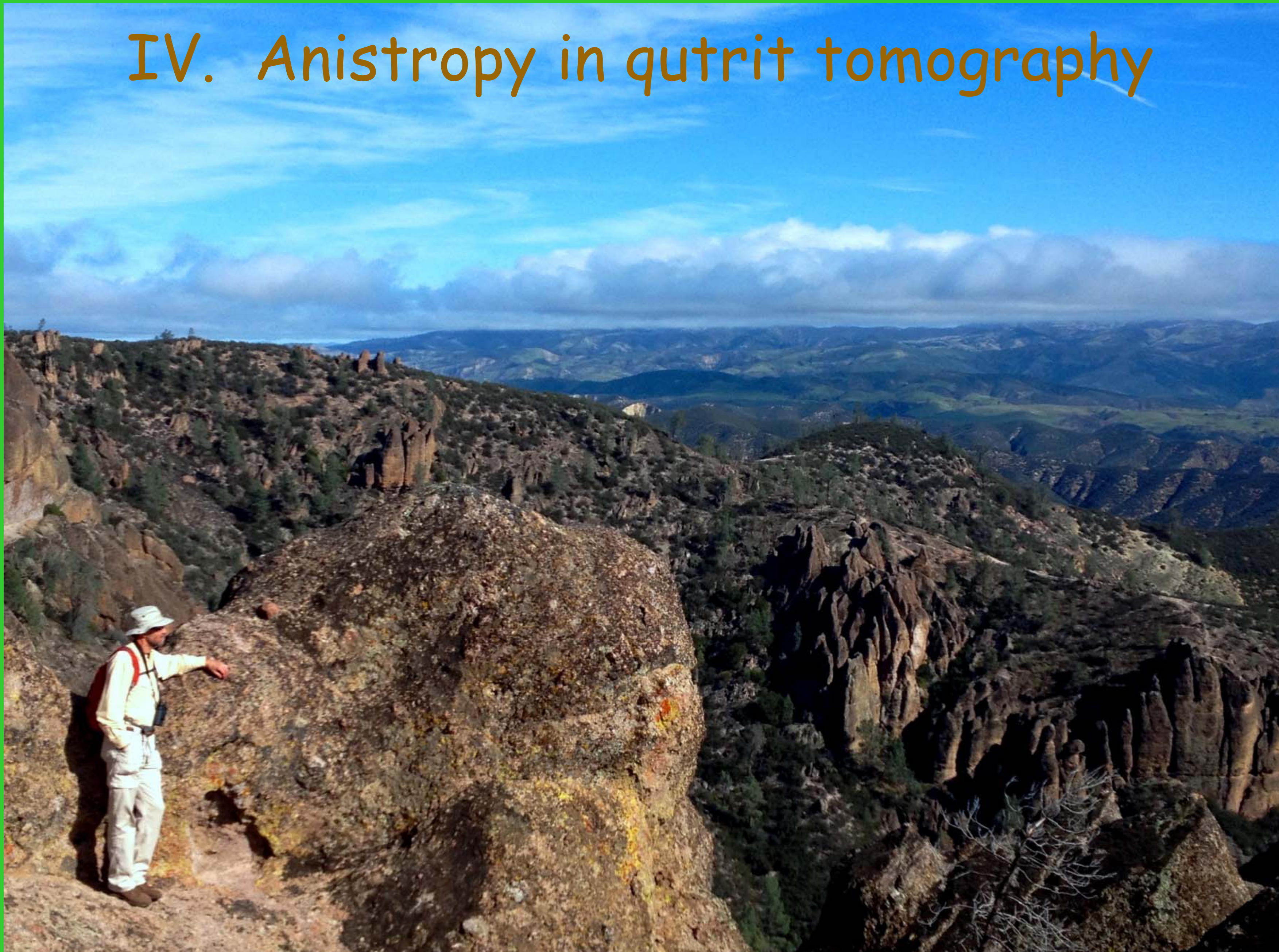
Are there spherical cows?

Can the Hilbert-Schmidt metric be the Fisher metric for a rank-one POVM?



Except for qubits and the maximally mixed state

IV. Anistropy in qutrit tomography



Pinnacles National Park
Central California

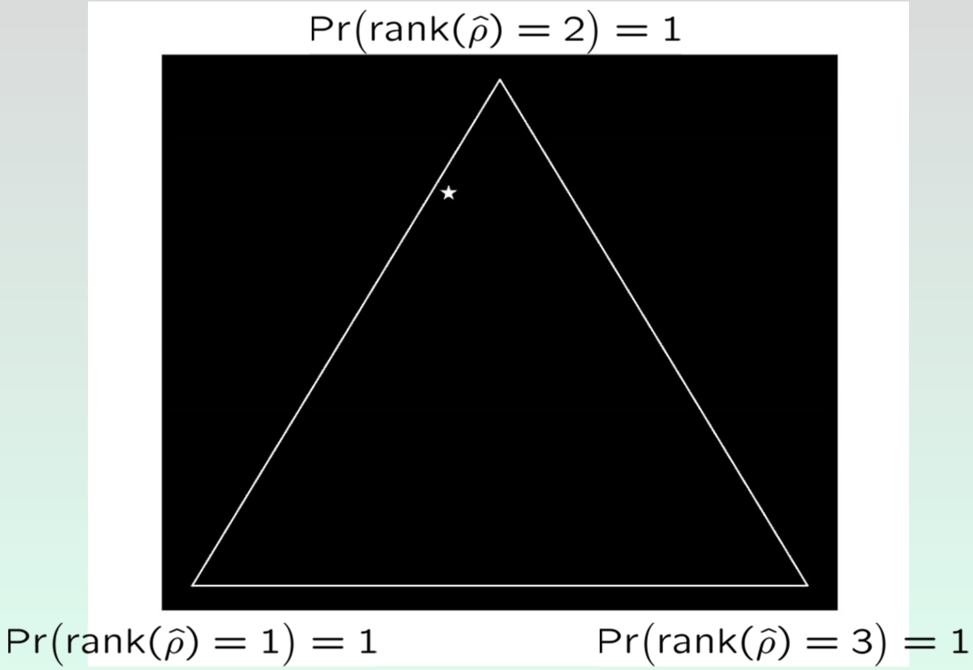
Anistropy in qutrit tomography
Rank distributions

For an isotropic distribution of estimates $\hat{\rho}$,

$$\begin{aligned}
\Pr\big(\mathrm{rank}(\hat{\rho}) = 3\big) &= \frac{1}{6} - \frac{\sqrt{3}}{4\pi} \simeq \frac{1}{35} \\
\Pr\big(\mathrm{rank}(\hat{\rho}) = 2\big) &= \frac{1}{2} + \frac{\sqrt{3}}{2\pi} \\
\Pr\big(\mathrm{rank}(\hat{\rho}) = 1\big) &= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \simeq \frac{1}{5}
\end{aligned}$$

Classical simplex

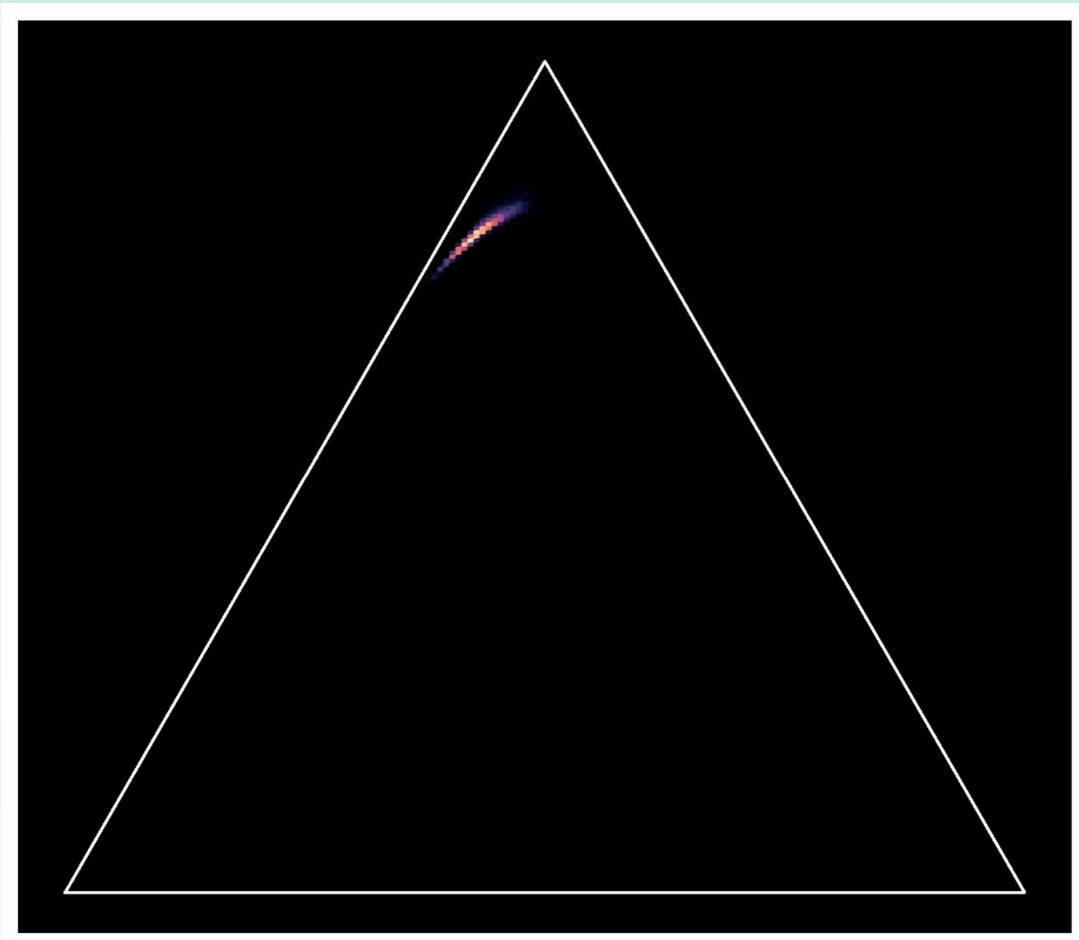
Quantum gerrymandering



Anistropy in qutrit tomography
Rank distributions

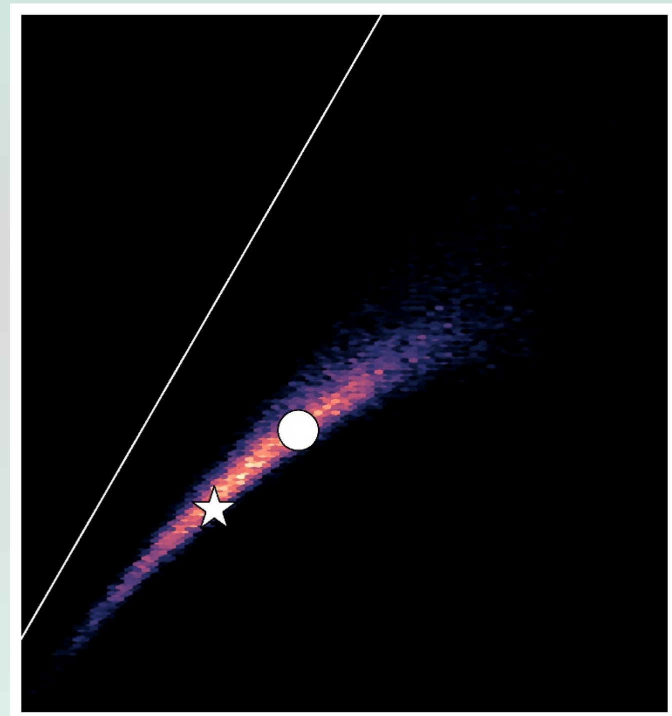
SIC-POVM

Rank 2



Rank 1

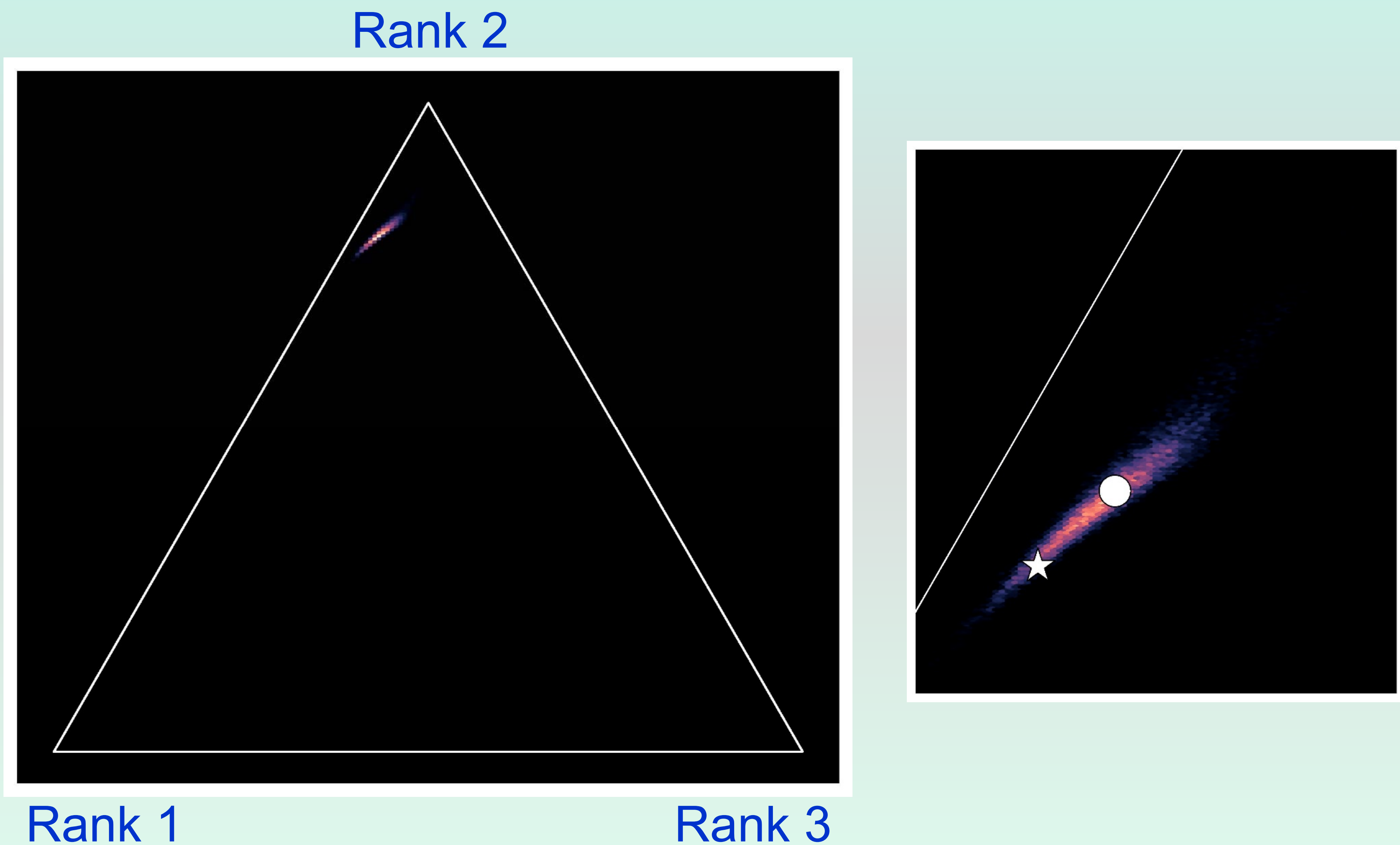
Rank 3



For the supercilious:
the supercilium

Anistropy in qutrit tomography
Rank distributions

MUB



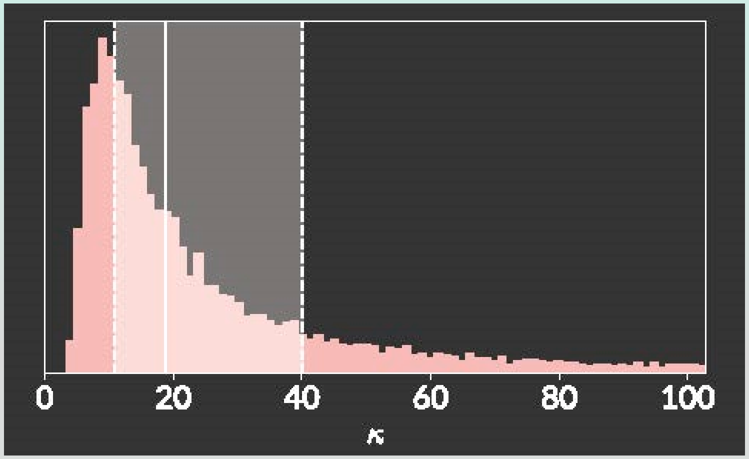
Anistropy in qutrit tomography

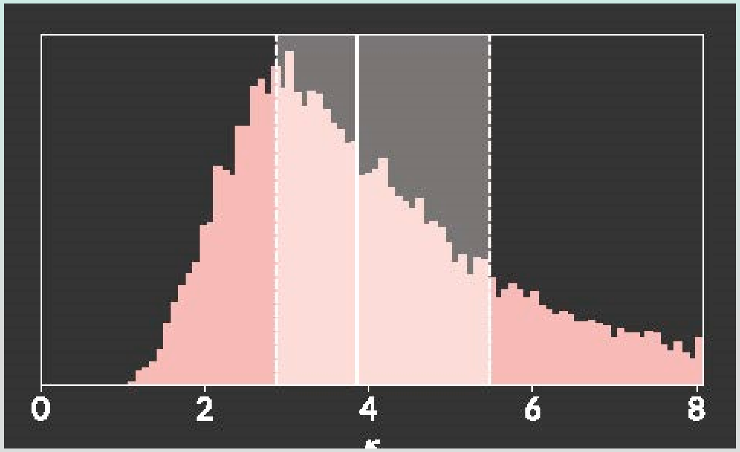
Condition numbers

Condition number: ratio of largest to smallest eigenvalue

SIC-POVM

relevant directions





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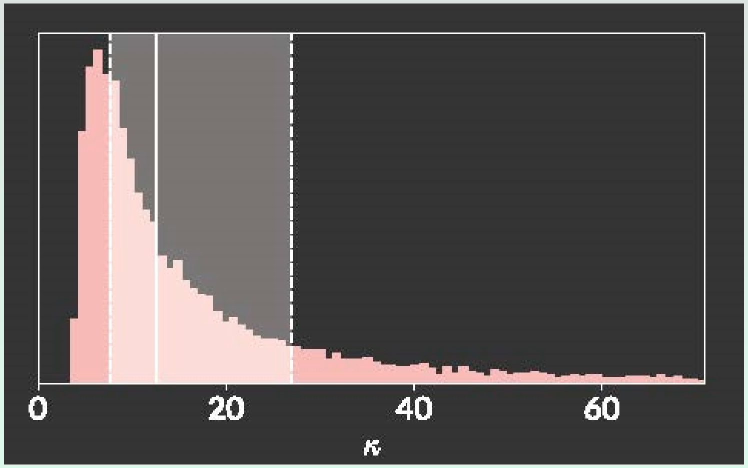

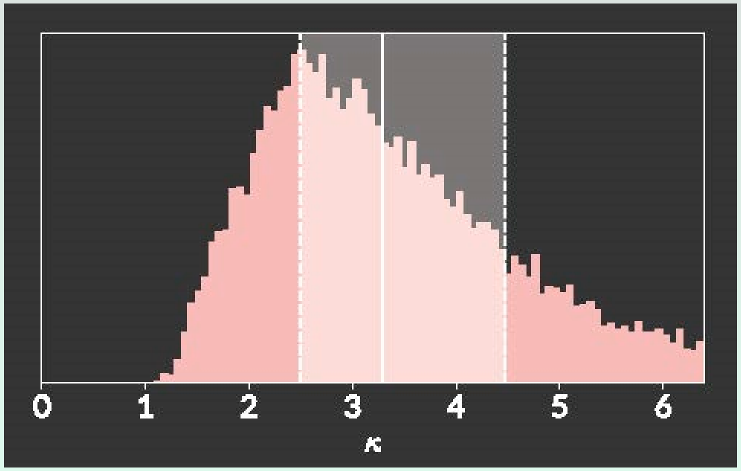

Dettifoss
Iceland