Quantum-limited measurements: One physicist's crooked path from quantum optics to quantum information

I. Introduction
II. Squeezed states and optical interferometry
III. Ramsey interferometry and cat states
IV. Quantum information perspective
V. Beyond the Heisenberg limit
VI. Conclusion

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Quantum circuits in this presentation were set using the LaTeX package Qcircuit, developed at the University of New Mexico by Bryan Eastin and Steve Flammia. The package is available at http://info.phys.unm.edu/Qcircuit/.
I. Introduction

In the Sawtooth Range
Central New Mexico
A new way of thinking

Quantum information science

Computer science
Computational complexity depends on physical law.

New physics
Quantum mechanics as liberator.
What can be accomplished with quantum systems that can’t be done in a classical world?
Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.
Quantum engineering

Old physics
Quantum mechanics as nag.
The uncertainty principle restricts what can be done.
Quantum information science

A way of thinking

Theory

Atomic, molecular, optical physics

Condensed-matter physics

Superconductivity physics

A way of doing

Experiment

Superconductivity physics

Metrology

Nuclear magnetic resonance (NMR)
Metrology
Taking the measure of things
The heart of physics

New physics
Quantum mechanics as liberator.
Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.
Quantum engineering

Old physics
Quantum mechanics as nag.
The uncertainty principle restricts what can be done.

Old conflict in new guise
II. Squeezed states and optical interferometry

Oljeto Wash
Southern Utah
(Really) high-precision interferometry

Laser Interferometer Gravitational Observatory (LIGO)

Hanford, Washington

Livingston, Louisiana

LIGO I

\[
\begin{pmatrix}
\text{differential strain} \\
\text{sensitivity}
\end{pmatrix} \approx 10^{-21}
\]

\[
\begin{pmatrix}
\text{differential displacement} \\
\text{sensitivity}
\end{pmatrix} \approx 4 \times 10^{-18} \text{ m}
\]

from 10 Hz to $10^3$ Hz.
(Really) high-precision interferometry

Hanford, Washington

Laser Interferometer Gravitational Observatory (LIGO)

Livingston, Louisiana

\[
\begin{align*}
\text{(differential strain sensitivity)} & \simeq 3 \times 10^{-23} \\
\text{(differential displacement sensitivity)} & \simeq 10^{-20} \text{ m}
\end{align*}
\]

from 10 Hz to 10^3 Hz.

High-power, Fabry-Perot cavity (multipass), recycling, squeezed-state (?) interferometers

LIGO II
Mach-Zender interferometer


\[
\begin{align*}
\Delta \phi &= \frac{\text{(noise)}}{|d(\text{signal})/d\phi|} = \frac{\sqrt{2} \Delta x_2}{\sqrt{N}} \\
\text{(signal)} &= N \cos \phi \\
\text{(noise)} &= \sqrt{2N} \Delta x_2 |\sin \phi|
\end{align*}
\]
Squeezed states of light

\[ \Delta \phi \sim \frac{\Delta x_2}{A} = \frac{\Delta x_2}{\sqrt{N}} \]

\[ \Delta x_1 = \Delta x_2 = \frac{1}{\sqrt{2}}, \quad \Delta \phi \sim \frac{1}{\sqrt{2N}} \]

\[ \Delta x_1 = e^r / \sqrt{2} \quad \Delta x_2 = e^{-r} / \sqrt{2}, \quad \Delta \phi \sim \frac{e^{-r}}{\sqrt{2N}} \]
Squeezed states of light

ANU Group continues development of squeezing for LIGO II and III.

Quantum limits on interferometric phase measurements

\[ \Delta \phi = \frac{\sqrt{2} \Delta x_2}{\sqrt{N}} \]

**Standard Quantum Limit (Shot-Noise Limit)**

\[ \Delta x_1 = \Delta x_2 = \frac{1}{\sqrt{2}}, \quad \Delta \phi = \frac{1}{\sqrt{N}} \]

**Heisenberg Limit**

\[ \frac{1}{2} (\Delta x_1)^2 \sim N \]

\[ \Delta x_2 = \frac{1}{2 \Delta x_1} = \frac{1}{2 \sqrt{2N}}, \quad \Delta \phi = \frac{1}{2N} \]

As much power in the squeezed light as in the main beam
III. Ramsey interferometry and cat states

Truchas from East Pecos Baldy
Sangre de Cristo Range
Northern New Mexico
Ramsey interferometry

\[ H = \frac{\hbar}{2} \omega \sigma_x + \frac{1}{2} e^{i\omega T} \sigma_y + \frac{1}{2} e^{-i\omega T} \sigma_x + \frac{1}{2} e^{i\omega T} \sigma_y + \frac{1}{2} e^{-i\omega T} \sigma_x + \frac{1}{2} e^{i\omega T} \sigma_y \]

\[ \rho_+ = \sin^2(\omega T) \]

\[ \rho_- = \cos^2(\omega T) \]

\[ \frac{1}{\sqrt{N}} \left| \frac{d(\text{signal})}{d(\omega T)} \right| = \Delta(\omega T) \]

\[ \langle \sigma_z \rangle = -\cos \omega T \]

\[ \Delta \sigma_z = \sqrt{1 - \cos^2 \omega T} = |\sin \omega T| \]

Frequency measurement

Time measurement

Clock synchronization

\( N \) independent “atoms”
Cat-state Ramsey interferometry


Fringe pattern with period $2\pi/N$

\[ (\text{signal}) = \langle \sigma_z \rangle = -\cos N\omega T \]
\[ (\text{noise}) = \Delta \sigma_z = \sqrt{1 - \cos^2 N\omega T} = |\sin N\omega T| \]

\[ \Delta(\omega T) = \frac{1}{\sqrt{\nu}} \frac{(\text{noise})}{|d(\text{signal})/d(\omega T)|} = \frac{1}{\sqrt{\nu}} \frac{1}{N} \]

\[ \nu = \text{(number of trials)} \]

$N$ cat-state atoms
Optical interferometry  Ramsey interferometry

$$\Delta \phi \sim \frac{1}{\sqrt{N}}$$

Standard Quantum Limit (Shot-Noise Limit)

Heisenberg Limit

$$\Delta \phi \sim \frac{1}{N}$$

$$\phi = \omega T$$

$$\Delta \phi \sim \frac{1}{N}$$

Something’s going on here.
Optical interferometry

\[ \Delta \phi \sim \frac{1}{\sqrt{N}} \]

Standard Quantum Limit (Shot-Noise Limit)

\[ \Delta \phi \sim \frac{1}{\sqrt{N}} \]

Heisenberg Limit

\[ \Delta \phi \sim \frac{1}{N} \]

Entanglement?
Between arms  Between atoms

\[ \sim 2r \text{ e-bits} \quad 1 \text{ e-bit} \]

Something's going on here.
IV. Quantum information perspective

Cable Beach
Western Australia
Quantum information version of interferometry

Standard quantum limit

Heisenberg limit

Fringe pattern with period $2\pi/N$

$|0\rangle \quad H \quad U_\phi = e^{-iZ\phi/2} \quad H \quad M$

$|0\rangle \quad H \quad U_\phi = e^{-iZ\phi/2} \quad H \quad M$

$|0\rangle \quad H \quad U_\phi = e^{-iZ\phi/2} \quad H \quad M$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\frac{1}{\sqrt{2}}(e^{-i\phi/2}|0\rangle + e^{i\phi/2}|1\rangle)$

$\cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle)$

$\frac{1}{\sqrt{2}}(e^{-iN\phi/2}|000\rangle + e^{iN\phi/2}|111\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|00\rangle$\quad $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

$N = 3$

Heisenberg limit

$|0\rangle \quad H \quad U_\phi \quad H \quad M$

$|0\rangle \quad U_\phi \quad U_\phi \quad U_\phi \quad H \quad M$

$[\cos(N\phi/2)|0\rangle - i\sin(N\phi/2)|1\rangle]|00\rangle$
Cat-state interferometer

State preparation

\[ U = e^{-i\phi} \]
\[ h = \sum_{j=1}^{N} h_j \]

Measurement

Single-parameter estimation

ancilla

Cat-state interferometer

\[ \left| 0 \rightangle \rightarrow H \rightarrow U_\phi \rightarrow H \rightarrow M \]

\[ \left| 0 \rightangle \rightarrow U_\phi \rightarrow U_\phi \rightarrow H \rightarrow M \]

\[ U_\phi = e^{-i\phi_1} \]
\[ U_\phi = e^{-i\phi_2} \]
\[ U_\phi = e^{-i\phi_3} \]
Heisenberg limit

\[ U_\phi = e^{-i\phi \mathbf{1}} \]
\[ U_\phi = e^{-i\mathbf{h}_1 \phi} \]
\[ U_\phi = e^{-i\mathbf{h}_2 \phi} \]
\[ U_\phi = e^{-i\mathbf{h}_3 \phi} \]

\[ U = e^{-i\mathbf{h} \phi} \]
\[ h = \sum_{j=1}^{N} h_j \]

\[ \Delta \phi \geq \frac{1}{2\Delta h} \geq \frac{1}{N(\Lambda - \lambda)} \]

\[ \Delta h \leq \frac{1}{2} \| h \| = \frac{1}{2} N(\Lambda - \lambda) \]

Separable inputs
\[ \Delta h \leq \frac{1}{2} \sqrt{N(\Lambda - \lambda)} \]
\[ \Delta \phi \geq \frac{1}{\sqrt{N(\Lambda - \lambda)}} \]

Achieving the Heisenberg limit

\[ |S\rangle \quad V \quad U_\phi = e^{-i\hbar_1 \phi} \quad W \quad M \]

\[ |S\rangle \quad U_\phi = e^{-i\hbar_2 \phi} \quad M \]

\[ |S\rangle \quad U_\phi = e^{-i\hbar_3 \phi} \quad M \]

\[ \text{cat state} \quad \frac{1}{\sqrt{2}} (|\Lambda, \ldots, \Lambda\rangle + |\lambda, \ldots, \lambda\rangle) \]

\[ \frac{1}{\sqrt{2}} \left( e^{-iN\Lambda \phi} |\Lambda, \ldots, \Lambda\rangle + e^{-iN\lambda \phi} |\lambda, \ldots, \lambda\rangle \right) \]

\[ e^{-iN(\Lambda+\lambda)\phi/2} \left( \cos[N(\Lambda - \lambda)\phi/2]|\Lambda, \ldots, \Lambda\rangle - i \sin[N(\Lambda - \lambda)\phi/2]|\lambda, \ldots, \lambda\rangle \right) \]

\[ \Delta \phi = \frac{1}{N(\Lambda - \lambda)} \]

Fringe pattern with period \(2\pi/N(\Lambda - \lambda)\)
Is it entanglement? Or not?

But what about?

- Optimal state for optical interferometry, the optical cat state \((|N, 0\rangle + |0, N\rangle)/\sqrt{2}\), called a "N00N" state, does a tiny bit better than inputting the optimal squeezed (Gaussian) state, but has just 1 e-bit of entanglement between the two arms, compared with the much larger \(\sim \log N\) e-bits of entanglement when inputting the optimal squeezed state.

- Flip half the spins in a cat state, and you get a state with the same amount of entanglement, but one that is worthless for metrology.

- Measurement sensitivity and optimal initial state depends on local Hamiltonians \(h_j\), but entanglement measures are usually constructed to be independent of such mundane details.

Need a generalized notion of entanglement that includes information about the physical situation, particularly the relevant Hamiltonian.
V. Beyond the Heisenberg limit

Echidna Gorge
Bungle Bungle Range
Western Australia
Beyond the Heisenberg limit

The purpose of theorems in physics is to lay out the assumptions clearly so one can discover which assumptions have to be violated.
Improving the scaling with $\mathcal{N}$

Cat state does the job.

Nonlinear Ramsey interferometry

\[ \Delta \phi \geq \frac{1}{2 \Delta h} \geq \frac{1}{\|h\|} = \frac{1}{\mathcal{N}^k (\Lambda^k - \lambda^k)} \]

Metrologically relevant $k$-body coupling

\[ h = \left( \sum_{j=1}^{N} h_j \right)^k = \sum_{j_1, \ldots, j_k} h_{j_1} h_{j_2} \cdots h_{j_k} \]

\[ \|h\| = \mathcal{N}^k (\Lambda^k - \lambda^k) \]

Improving the scaling with $N$ without entanglement


Product input

$|S\rangle$

Product measurement

$|S\rangle$

$|S\rangle$

$U_\phi = e^{-i\hbar\phi}$

$\hbar = \left( \sum_{j=1}^{N} Z_j / 2 \right)^k = J_z^k$

$\Delta \phi \sim \frac{1}{N^{k-1/2}}$
Improving the scaling with $N$ without entanglement


$$h = \left( \sum_{j=1}^{N} \frac{Z_j}{2} \right)^k = J_z^k$$

$k = 2$

- Prepare system with all spins up.
- Rotate all spins by $\beta = \pi/4$ about $y$ axis.
- For short times, $\phi \ll 1/\sqrt{J}$, nonlinear coupling rotates spins with angular velocity $\langle J_z \rangle = 2J \cos \beta$ about $z$ axis.
- Measure equatorial component of $J$.
Improving the scaling with $N$ without entanglement

\[\frac{z}{2} \left( a^+ a + b^+ b \right) \right) |0\rangle \]

\[\frac{\phi (a^+ a)^2 + \Theta (b^+ b)^2}{\sqrt{N!}} \]

\[= (\phi + \Theta) (N^2/4 + J_z^2) + (\phi - \Theta) NJ_z \]
Improving the scaling with $N$ without entanglement

\[ \phi(a \dagger a)^2 + \Theta(b \dagger b)^2 = (\phi + \Theta)(N^2/4 + J_z^2) + (\phi - \Theta)NJ_z \]
VI. Conclusion

Bungle Bungle Range
Western Australia
Making quantum limits relevant: One metrology story

Resources

- Overall measurement time $\tau$ (the "classical serial resource")
- Interaction time $T$ for an individual probe (the "quantum serial resource")
- Rate $R$ at which systems can be deployed [$R(T - T) = n$ is the "classical parallel resource"]
- Entanglement within each probe consisting of $N$ systems ($N$ is the "quantum parallel resource")

Problem

Given $\tau$ and $R$ and a decoherence rate $\Gamma$, what is the best strategy for estimating frequency $\omega = \phi/T$?

The answer is well known for optical interferometry and has recently been analyzed in detail for linear Ramsey interferometry: The quantum resources—extended coherent evolution and entanglement—are useful only if $\Gamma \tau \ll 1$ and $R \tau \gg 1$.

Making quantum limits relevant:
One metrology story

<table>
<thead>
<tr>
<th>Range of $\Gamma_\tau$</th>
<th>$T$</th>
<th>$N$</th>
<th>$\nu$</th>
<th>$n$</th>
<th>$\delta\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\nu_{\text{min}}\Gamma}{R} \leq \Gamma_\tau &lt; \frac{2\nu_{\text{min}}\Gamma}{R}$</td>
<td>$T_s$</td>
<td>1</td>
<td>$\nu_{\text{min}}$</td>
<td>$\nu_{\text{min}}$</td>
<td>$\frac{e^{\Gamma T_s}}{T_s\sqrt{\nu_{\text{min}}}}$</td>
</tr>
<tr>
<td>$\frac{2\nu_{\text{min}}\Gamma}{R} \leq \Gamma_\tau &lt; \sqrt{\frac{2\nu_{\text{min}}\Gamma}{R}}$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{R\tau}{2\nu_{\text{min}}}$</td>
<td>$\nu_{\text{min}}$</td>
<td>$\frac{R\tau}{2}$</td>
<td>$\frac{4\sqrt{\nu_{\text{min}}}}{R\tau^2}e^{\Gamma R\tau^2/4\nu_{\text{min}}}$</td>
</tr>
<tr>
<td>$\sqrt{\frac{2\nu_{\text{min}}\Gamma}{R}} \leq \Gamma_\tau &lt; 1$</td>
<td>$\frac{\tau}{2}$</td>
<td>$\frac{1}{\Gamma_\tau}$</td>
<td>$\frac{\Gamma R\tau^2}{2}$</td>
<td>$\frac{R\tau}{2}$</td>
<td>$\frac{2\sqrt{2e}}{\tau\sqrt{R/\Gamma}}$</td>
</tr>
<tr>
<td>$\Gamma_\tau \geq 1$</td>
<td>$T_p$</td>
<td>1</td>
<td>$R(\tau - T_p)$</td>
<td>$R(\tau - T_p)$</td>
<td>$\frac{e^{\Gamma T_p}}{T_p\sqrt{R(\tau - T_p)}}$</td>
</tr>
</tbody>
</table>

$T_s = \tau - \nu_{\text{min}}/R$

$T_p = \frac{3/2 + \Gamma - \sqrt{(3/2 + \Gamma)^2 - 4\Gamma\tau}}{2\Gamma}$
Using quantum circuit diagrams

Cat-state interferometer

|0⟩ → H → $e^{-iZ\phi/2}$ → H → M

|0⟩ → Z

|0⟩ → X → Z → X

|0⟩ → Z → -I → Z

|0⟩ → $e^{-iZ\otimes Z\phi/2}$ → $e^{-iZ\phi/2}$

Cat-state interferometer

|0⟩ → H → $e^{-3iZ\phi/2}$ → H → M

|0⟩ → Z

|0⟩ → X → Z → X

|0⟩ → Z → -I → Z

|0⟩ → $e^{-iZ\otimes Z\phi/2}$ → $e^{-iZ\phi/2}$