Quantum limits on estimating a waveform

I. Introduction. What's the problem?
II. Standard quantum limit (SQL) for force detection. The right wrong story
III. Beating the SQL. Three strategies

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I. Introduction. What's the problem?

View from Cape Hauy
Tasman Peninsula
Tasmania

Measuring a classical parameter

Phase shift in an (optical) interferometer

- Readout of anything that changes optical path lengths
- Michelson-Morley experiment
- Gravitational-wave detection
- Planck-scale, holographic uncertainties in positions

Torque on or free precession of a collection of spins

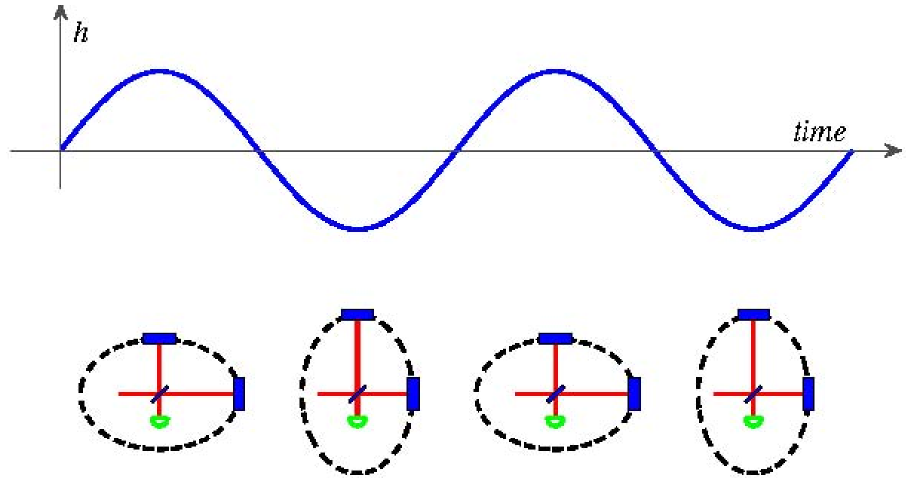
- Magnetometer
- Atomic clock

Force on a linear system

- Gravitational-wave detection
- Accelerometer
- Gravity gradiometer
- Electrometer
- Strain meter

(Absurdly) high-precision interferometry for force sensing

Hanford, Washington



The LIGO Collaboration, Rep. Prog. Phys. 72, 076901 (2009).

Laser Interferometer Gravitational Observatory (LIGO)

Livingston, Louisiana

(Absurdly) high-precision interferometry for force sensing

Hanford, Washington

Initial LIGO

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-21}$$

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 4 \times 10^{-18}\,\mathrm{m}$$

from $40\,\mathrm{Hz}$ to $7{,}000\,\mathrm{Hz}$.

Laser Interferometer Gravitational Observatory (LIGO)

High-power, Fabry-Perot-cavity (multipass), power-recycled interferometers

Livingston, Louisiana

(Absurdly) high-precision interferometry for force sensing

Advanced LIGO

Hanford, Washington

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 3\times 10^{-23}$$

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-19}\,\text{m}$$

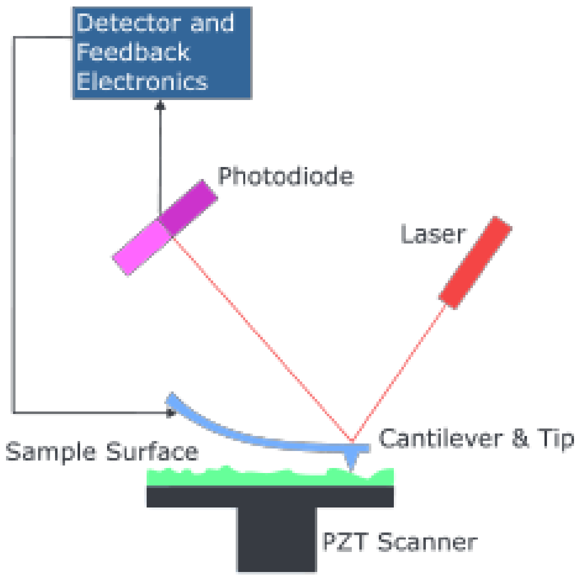
from $10\,\text{Hz}$ to $7{,}000\,\text{Hz}$.

Laser Interferometer Gravitational Observatory (LIGO)

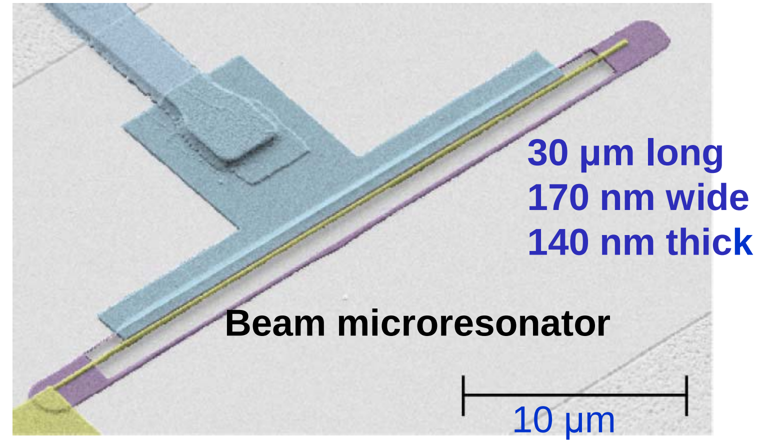
Livingston, Louisiana

High-power, Fabry-Perot-cavity (multipass), power- and signal-recycled, squeezed-light interferometers

Opto,atomic,electro micromechanics

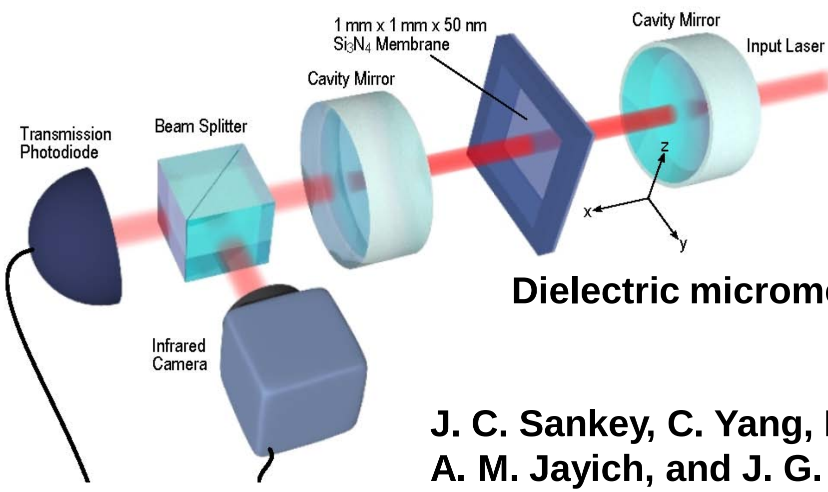


Atomic force microscope



Beam microresonator

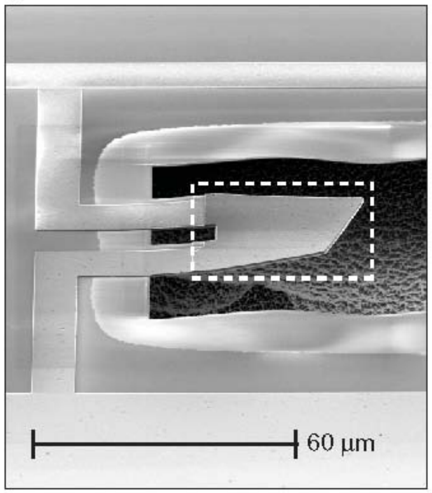
T. Rocheleau, T. Ndukum, C. Macklin, J. B. Hertzberg, A. A. Clerk, and K. C. Schwab, Nature 463, 72 (2010).

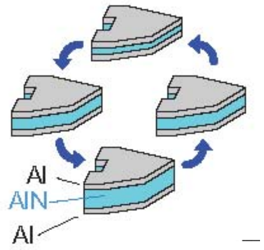


Dielectric micromembrane

J. C. Sankey, C. Yang, B. M. Zwickl, A. M. Jayich, and J. G. E. Harris, Nature Physics 6, 707 (2010).

Opto,atomic, electro micromechanics

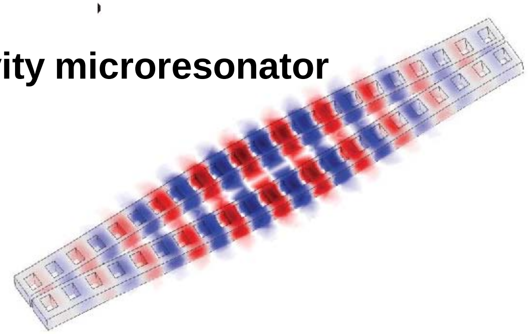


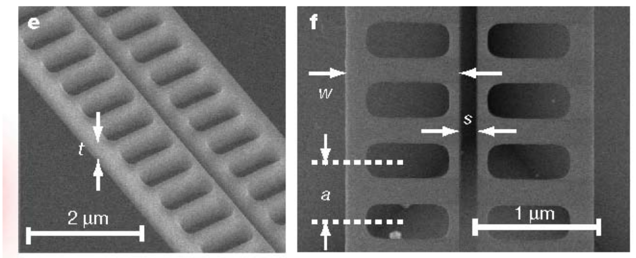


Drum microresonator

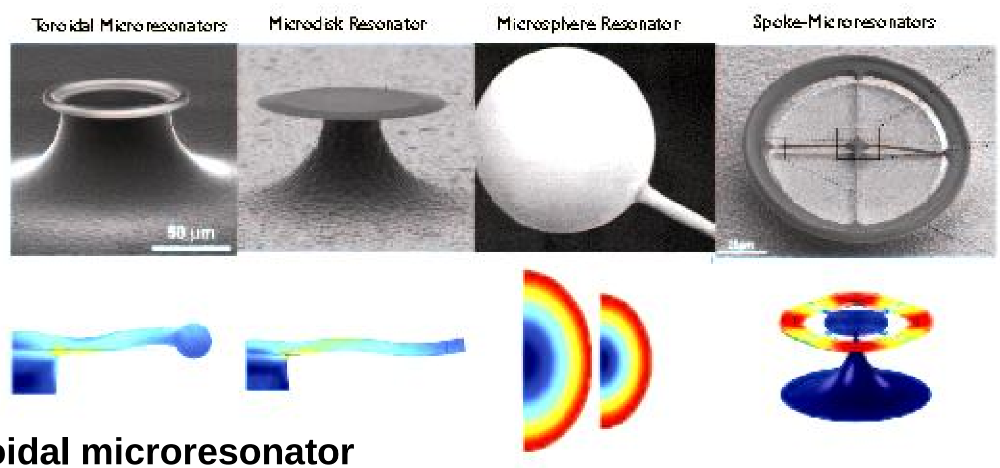
A. D. O'Connell *et al.*, Nature 464, 697 (2010).

Zipper-cavity microresonator





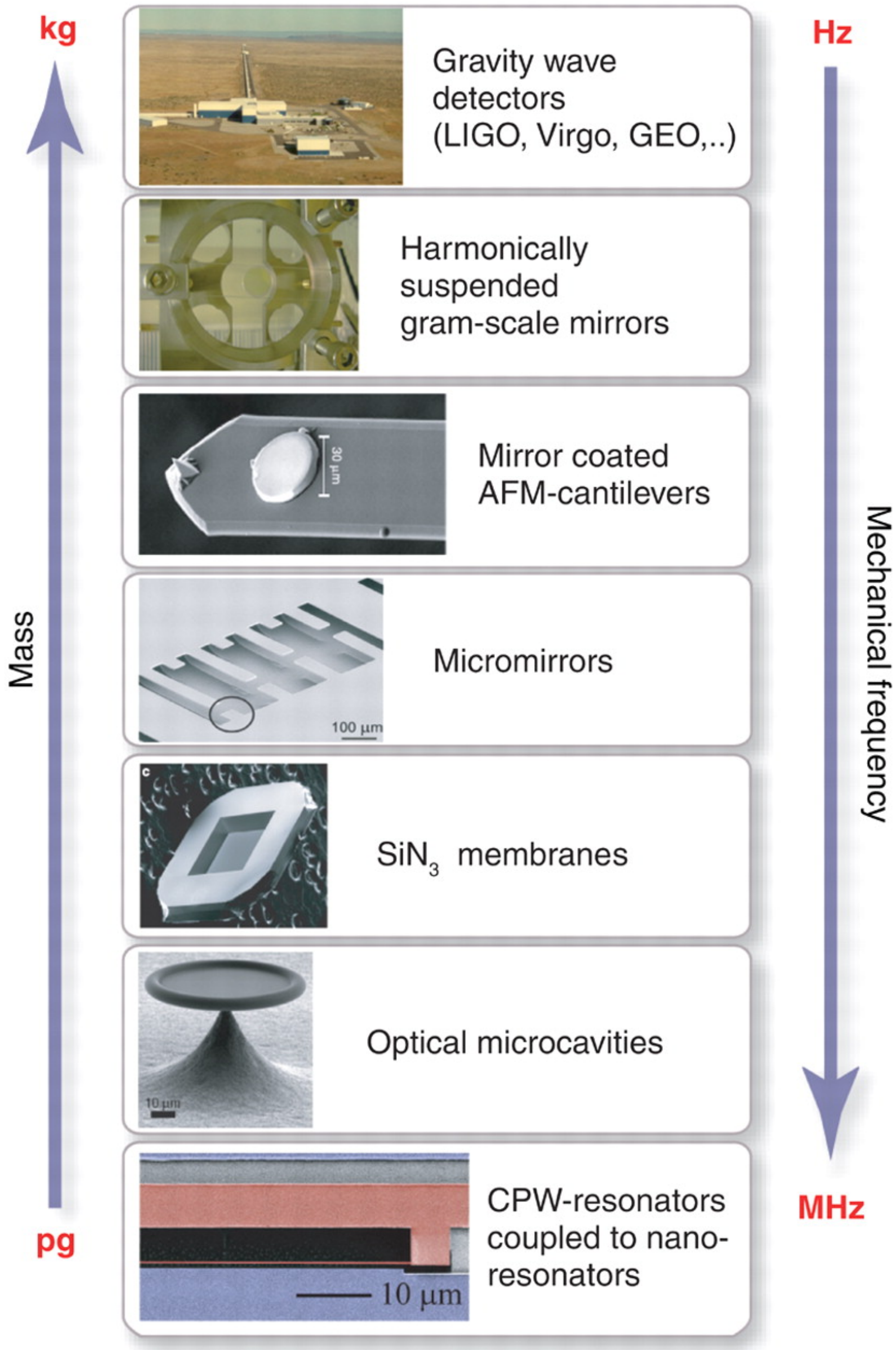
M. Eichenfield, R. Camacho, J. Chan, K. J. Vahala, and O. Painter, Nature 459, 550 (2009).



Toroidal microresonator

A. Schliesser and T. J. Kippenberg, Advances in Atomic, Molecular, and Optical Physics, Vol. 58, (Academic Press, San Diego, 2010), p. 207.

Mechanics for force sensing



T. J. Kippenberg and K. J. Vahala, Science 321, 172 (2008).

Science AAAS

Standard quantum limit (SQL)

Wideband detection of force *f* on free mass *m*

LIGO interferometer

$$\Delta q \simeq \sqrt{\Delta q_0^2 + \boxed{\frac{\Delta p_0^2 \tau^2}{m^2}}} \geq \sqrt{\frac{2\tau \Delta q_0 \Delta p_0}{m}} \geq \sqrt{\frac{\hbar \tau}{m}} \equiv \Delta q_{\mathrm{SQL}}$$

Back action

$$\delta q \simeq \frac{f\tau^2}{2m} \quad \Longrightarrow \quad f_{\mathrm{SQL}} \equiv \frac{2m}{\tau^2}\Delta q_{\mathrm{SQL}} = \sqrt{\frac{4\hbar m}{\tau^3}}$$

$$m \simeq 50\,\mathrm{kg}, \qquad \Delta\nu = 1/\tau \simeq 100\,\mathrm{Hz}$$

$$\Longrightarrow \quad \Delta q_{\mathrm{SQL}} \simeq 10^{-19}\,\mathrm{m}, \qquad f_{\mathrm{SQL}} \simeq 100\,\mathrm{fN}$$

Standard quantum limit (SQL)

Narrowband, on-resonance detection of force *f* on oscillator of mass *m* and resonant frequency ω_0

Nanoresonator

$$\Delta q_{\mathrm{SQL}} \equiv \sqrt{\frac{\hbar}{2m\omega_0}}$$

Back action?

$$\delta q \simeq \frac{f\tau}{2m\omega_0} \quad\Longrightarrow\quad f_{\mathrm{SQL}} \equiv \frac{2m\omega_0}{\tau}\Delta q_{\mathrm{SQL}} = \sqrt{\frac{2\hbar m\omega_0}{\tau^2}}$$

$$m \simeq 10\,\mathrm{pg}, \qquad 1/\tau_0 = \omega_0/2\pi \simeq 10\,\mathrm{MHz}, \qquad Q \simeq 10^4\text{–}10^6$$

$$\Longrightarrow \quad \Delta q_{\mathrm{SQL}} \simeq 10\,\mathrm{fm}, \qquad f_{\mathrm{SQL}} \simeq 100\,\mathrm{fN} \times \frac{\tau_0}{\tau}$$

$$\left(\begin{array}{c}\text{force between two}\\ \text{Bohr magnetons}\\ \text{separated by } r = 1\,\mathrm{nm}\end{array}\right) = \frac{\mu_0}{4\pi} \times \frac{\mu_B^2}{r^4} \simeq 10\,\mathrm{aN}$$

$$\mu_B = e\hbar/2m_e c = e\lambda_c/4\pi \simeq e \times 0.2\,\mathrm{pm}$$

SQL

Wideband force *f* on free mass *m*

$$\Delta q_{\mathrm{SQL}} = \sqrt{\frac{\hbar\tau}{m}} \qquad f_{\mathrm{SQL}} = \sqrt{\frac{4\hbar m}{\tau^3}} = \Delta\nu\sqrt{4\hbar m(\Delta\nu)}$$

On-resonance force *f* on oscillator of mass *m* and resonant frequency $\boldsymbol{\omega_0}$

$$\Delta q_{\mathrm{SQL}} = \sqrt{\frac{\hbar}{2m\omega_0}} \qquad f_{\mathrm{SQL}} = \sqrt{\frac{2\hbar m\omega_0}{\tau^2}} = \Delta\nu\sqrt{2\hbar m\omega_0}$$

It's wrong.

It's not even the right wrong story.

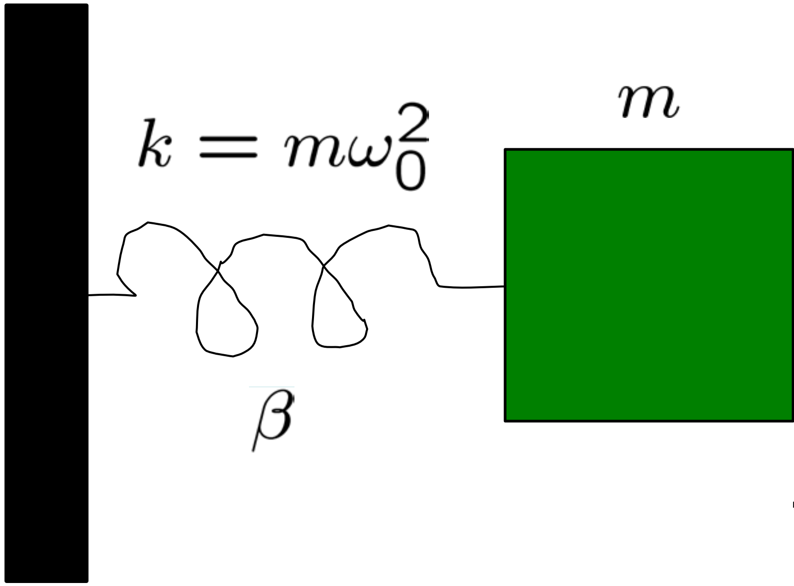
The right wrong story. Waveform estimation.

$$S_{\mathrm{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m\sqrt{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}$$

II. Standard quantum limit (SQL) for force detection. The right wrong story

Oljeto Wash
Southern Utah

SQL for force detection



Signal force $f(t)$ (waveform)

$\leftarrow$ coupling $-qf(t)$

$\leftarrow$ Langevin force $L(t)$

$\leftarrow$ Back-action force $\xi(t)$

$\longrightarrow q(t)$

$\downarrow$

Monitor position

Readout $y(t) = q(t) + \eta(t)$

Measurement noise

Back-action force

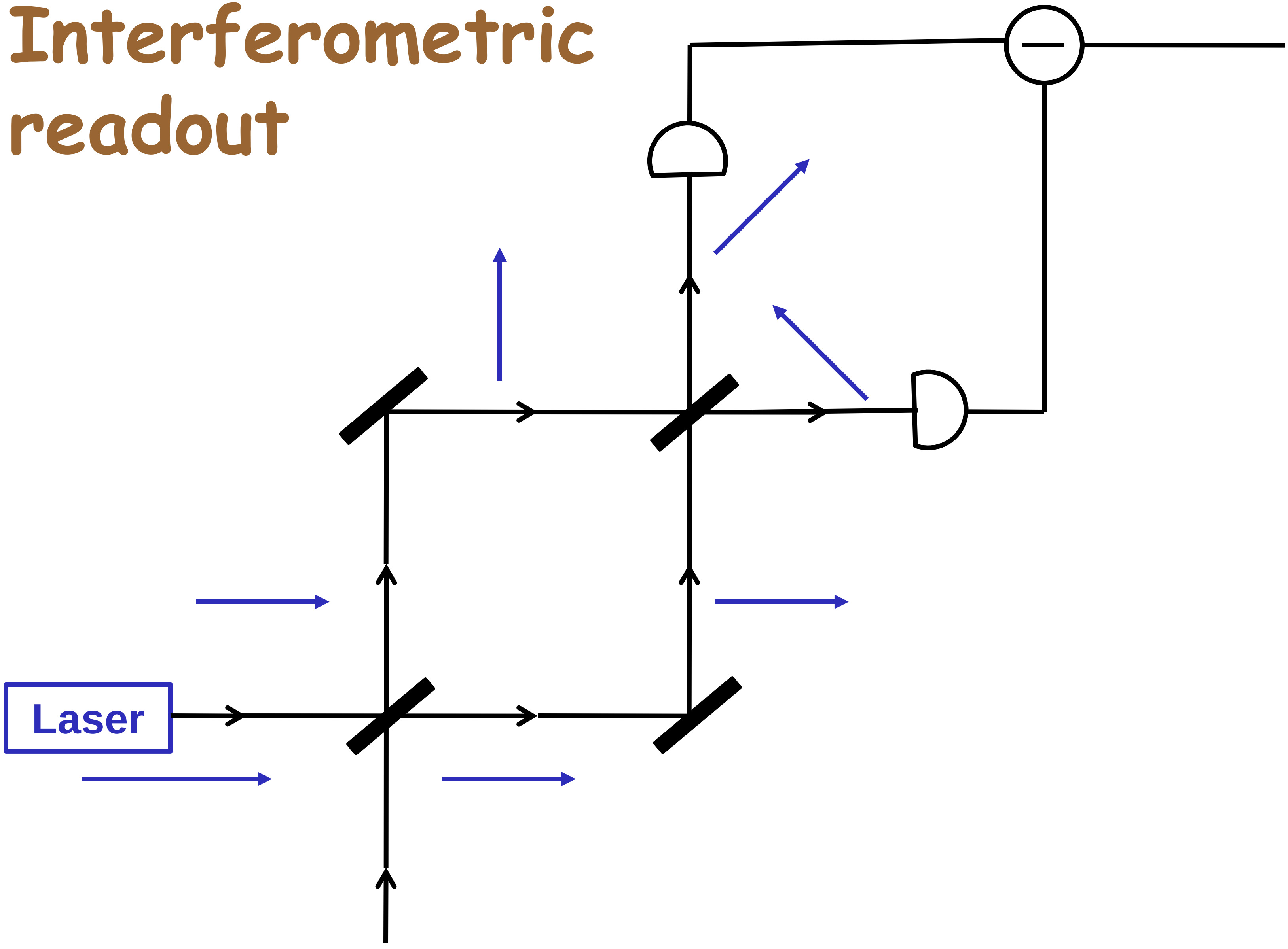
$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{f(t)}{m} + \frac{\boxed{\xi(t)}}{m} + \frac{\boxed{L(t)}}{m}$$

Langevin force

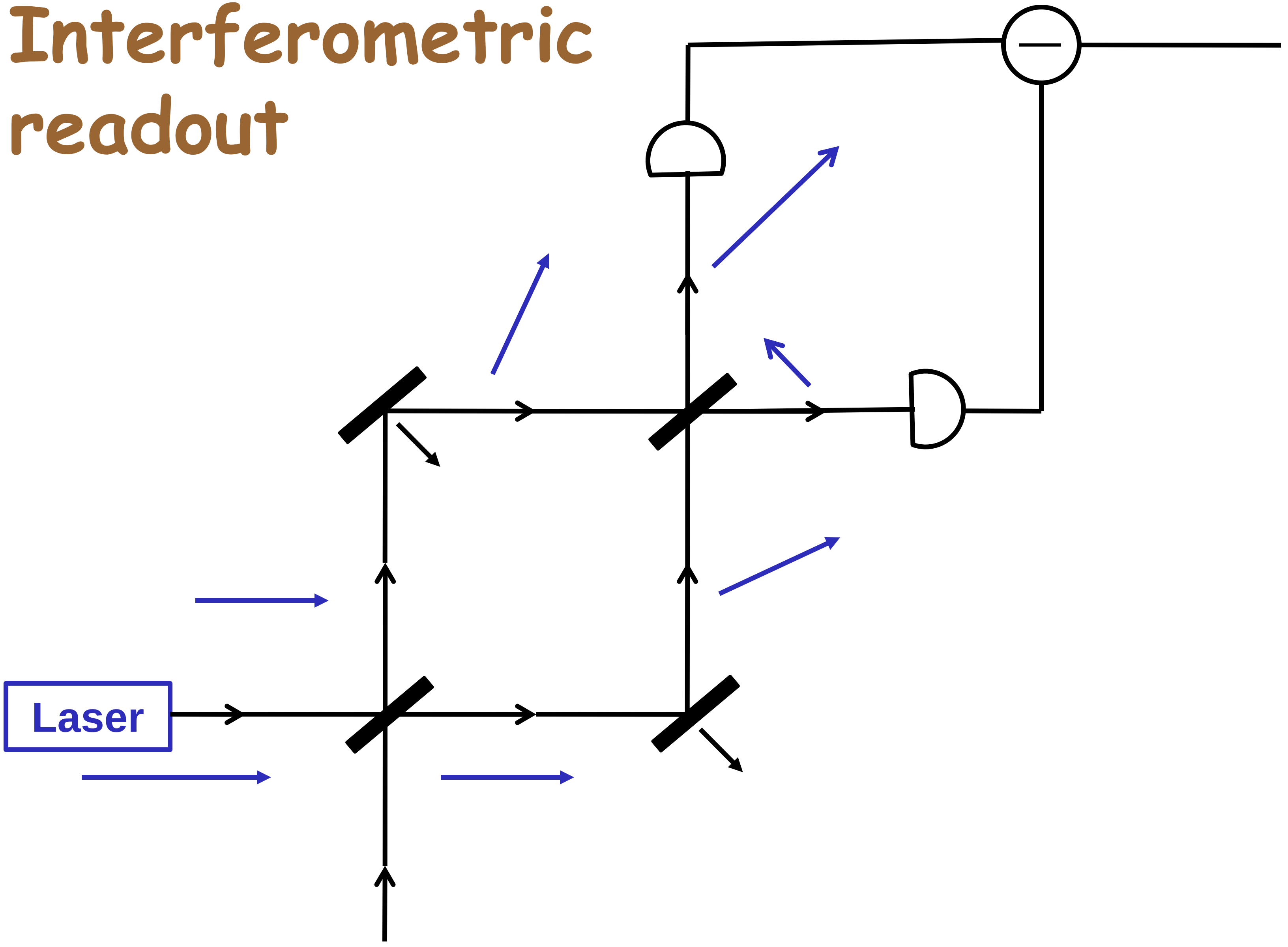
$$y(t) = q(t) + \boxed{\eta(t)}$$

measurement (shot) noise

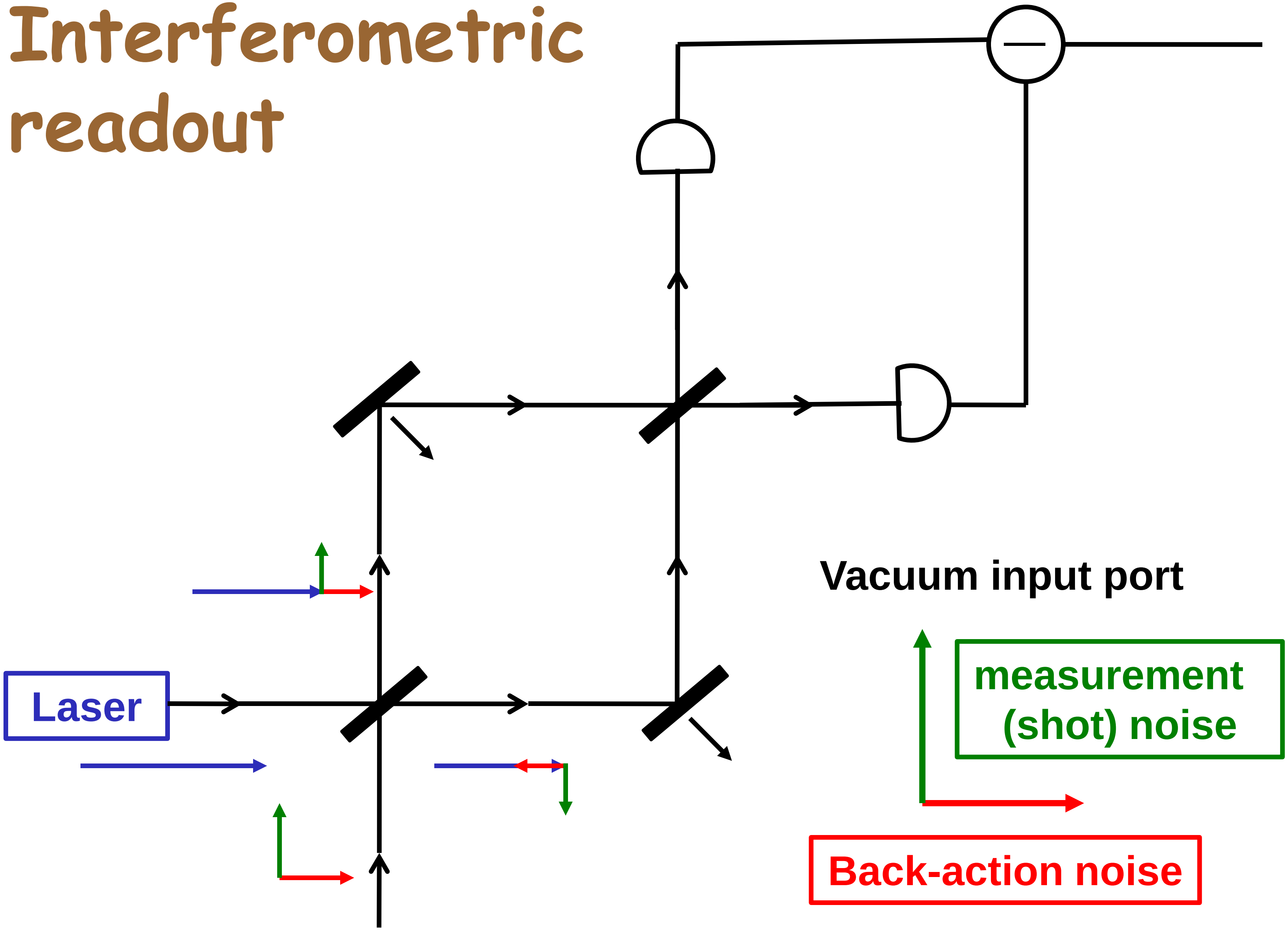
Interferometric readout



Interferometric readout

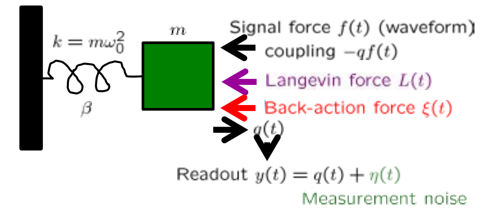


Interferometric readout



If shot noise dominates, squeeze the phase quadrature.

SQL for force detection



Time domain

Back-action force

Langevin force

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{f(t)}{m} + \frac{\boxed{\xi(t)}}{m} + \frac{\boxed{L(t)}}{m}$$

$$y(t) = q(t) + \boxed{\eta(t)}$$

measurement noise

$$q(\omega) = G(\omega)[f(\omega) + \xi(\omega) + L(\omega)]$$

Frequency domain

$$\left(\begin{array}{c}\text{response or}\\ \text{transfer function}\end{array}\right) = G(\omega) \equiv \frac{1}{m(\omega_0^2 - \omega^2 - 2i\beta\omega)}$$

Back-action force

$$z(\omega) = \frac{1}{G(\omega)} y(\omega) = f(\omega) + \boxed{\frac{\eta(\omega)}{G(\omega)}} + \boxed{\xi(\omega)} + \boxed{L(\omega)}$$

measurement noise

Langevin force

Noise-power spectral densities

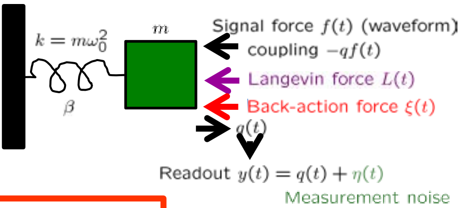
Zero-mean, time-stationary random process *u(t)*

$$u(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, u(\omega) e^{-i\omega t} \qquad u(\omega) = \int_{-\infty}^{\infty} dt\, u(t) e^{i\omega t}$$

Noise-power spectral density of *u*

$$\langle u^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S_u(\omega)$$

SQL for force detection



Back-action force

$$z(\omega) = \frac{1}{G(\omega)} y(\omega) = f(\omega) + \boxed{\frac{\eta(\omega)}{G(\omega)}} + \boxed{\xi(\omega)} + \boxed{L(\omega)}$$

measurement noise **Langevin force**

$$S_{\Delta z}(\omega) = \boxed{\frac{S_\eta(\omega)}{|G(\omega)|^2}} + \boxed{S_\xi(\omega)} + \boxed{S_L(\omega)}$$

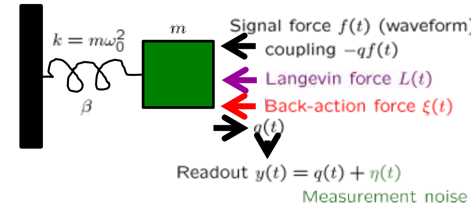
Fluctuation-dissipation theorem

$$S_L(\omega) = 4m\beta\hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1} \right)$$

Quantum mechanics of continuous measurement

$$S_\eta(\omega) S_\xi(\omega) \geq \hbar^2/4$$

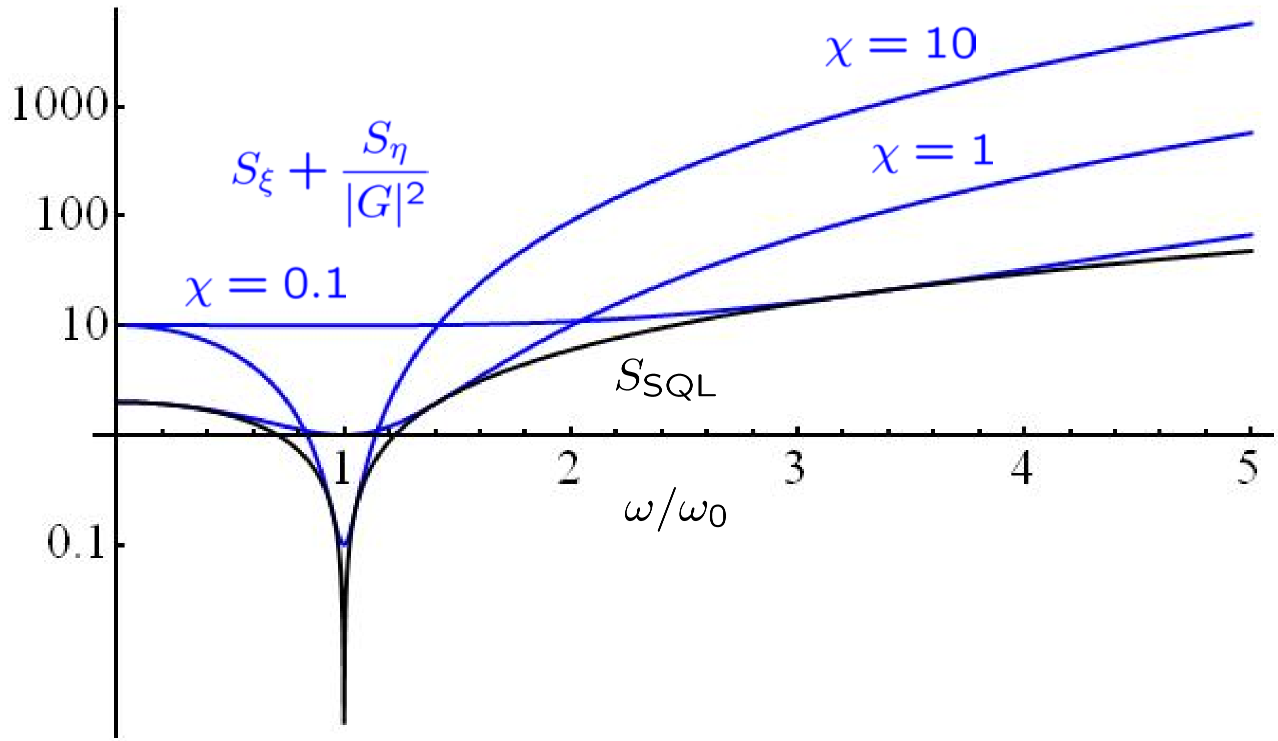
SQL for force detection



$$S_{\Delta z}(\omega) = \boxed{\frac{S_\eta(\omega)}{|G(\omega)|^2}} + \boxed{S_\xi(\omega)} \boxed{\geq} \frac{2\sqrt{S_\eta(\omega) S_\xi(\omega)}}{|G(\omega)|} \boxed{\geq} \frac{\hbar}{|G(\omega)|} \equiv S_{\mathsf{SQL}}(\omega)$$

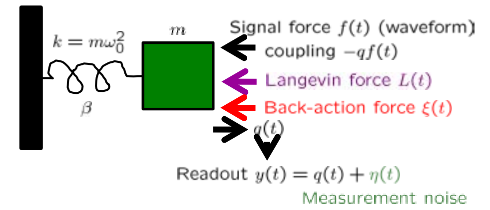
$$\boxed{= \iff S_\eta(\omega) = S_\xi(\omega)|G(\omega)|^2}$$

$$\boxed{= \iff S_\eta(\omega) S_\xi(\omega) = \hbar^2/4}$$



$$S_\eta S_\xi = \hbar^2/4 \qquad \frac{m^2 \omega_0^4 S_\eta}{S_\xi} \equiv \chi \qquad Q = \frac{\omega_0}{2\beta} = 10^3$$

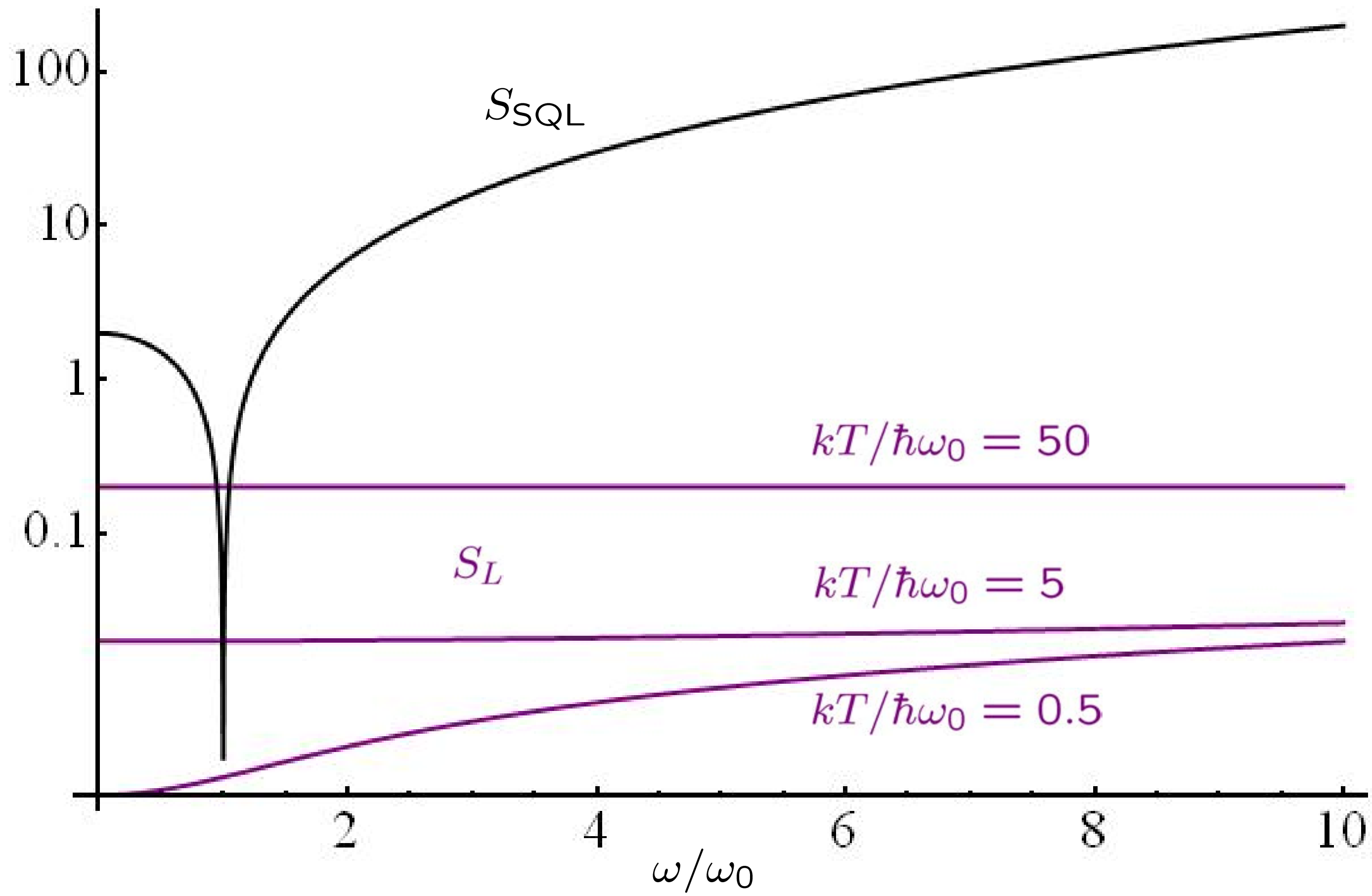
Langevin force



$$S_L(\omega) = 4m\beta\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1}\right)$$

$$S_{\mathrm{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}$$

$$Q = \frac{\omega_0}{2\beta} = 10^3$$



SQL for force detection

$$S_{\mathsf{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}$$

The right wrong story.

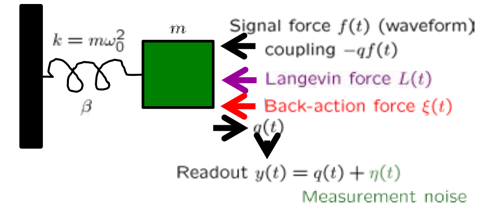
$$\frac{S_\eta(\omega)}{S_\xi(\omega)} = |G(\omega)|^2 \qquad S_\eta(\omega) S_\xi(\omega) = \hbar^2/4$$

In an opto-mechanical setting, achieving the SQL at a particular frequency requires squeezing at that frequency, and achieving the SQL over a wide bandwidth requires frequency-dependent squeezing.

III. Beating the SQL. Three strategies

Truchas from East Pecos Baldy
Sangre de Cristo Range
Northern New Mexico

Beating the SQL. Strategy 1



1. Couple parameter to observable *h*, and monitor observable *o* conjugate to *h.*
2. Arrange that *h* and *o* are *conserved* in the absence of the parameter interaction; *o* is the simplest sort of *quantum nondemolition* (QND) or *back-action-evading* (BAE) observable.
3. Give *o* as small an uncertainty as possible, thereby giving *h* as big an uncertainty as possible (back action).

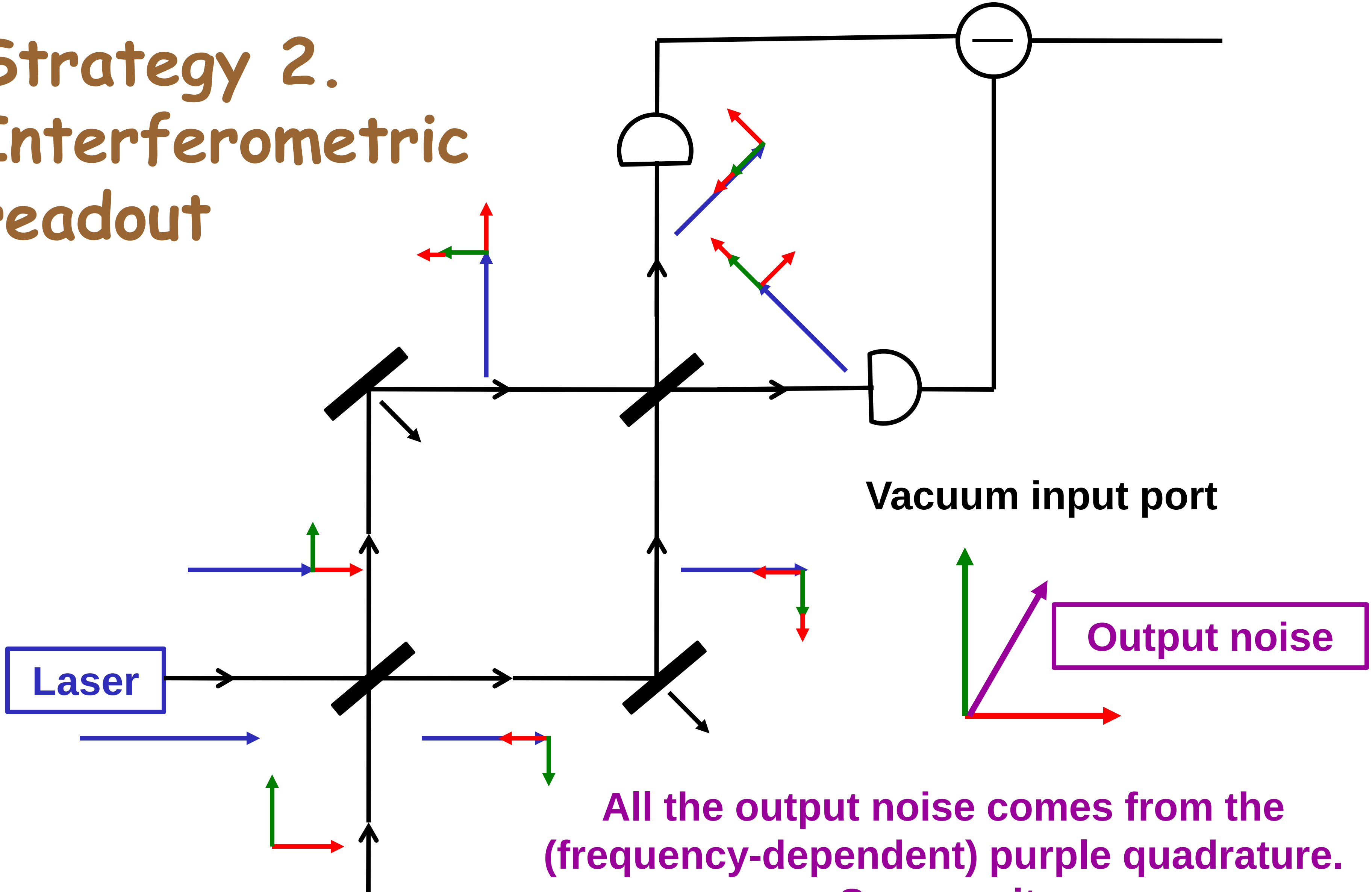
Strategy 1. Monitor a quadrature component.

$$\begin{aligned} q &= \mathrm{Re}[(X_1 + iX_2)e^{-i\omega_0 t}] = X_1 \cos\omega_0 t + X_2 \sin\omega_0 t \\ p/m\omega_0 &= \mathrm{Im}[(X_1 + iX_2)e^{-i\omega_0 t}] = -X_1 \sin\omega_0 t + X_2 \cos\omega_0 t \end{aligned}$$

Downsides

1. **Detect only one quadrature of the force.**
2. **Mainly narrowband (no convenient free-mass version).**
3. **Need new kind of coupling to monitor oscillator.**

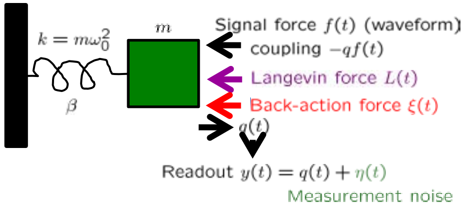
Strategy 2. Interferometric readout



All the output noise comes from the (frequency-dependent) purple quadrature. Squeeze it.

W. G. Unruh, in Quantum Optics, Experimental Gravitation, and Measurement Theory, edited by P. Meystre and M. O. Scully (Plenum, 1983), p. 647; F. Ya. Khalili, PRD 81, 122002 (2010).

Beating the SQL. Strategy 2



Strategy 2. Squeeze the entire output noise by correlating the measurement and back-action noise.

$$y(\omega) = \underbrace{G(\omega)f(\omega) + G(\omega)\xi(\omega)}_{= q(\omega)} + \eta(\omega)$$

Squeeze this output noise by correlating η and ξ. Quantum mechanics requires that an orthogonal linear combination of η and $G\xi$ become very noisy, thus making η, ξ, *and* q very noisy.

Quantum Cramér-Rao Bound (QCRB)

Single-parameter estimation: Bound on the error in estimating a classical parameter that is coupled to a quantum system in terms of the inverse of the quantum Fisher information.

Multi-parameter estimation: Bound on the covariance matrix in estimating a set of classical parameters that are coupled to a quantum system in terms of the inverse of a quantum Fisher-information matrix.

Waveform estimation: Bound on the continuous covariance matrix for estimating a continuous waveform that is coupled to a quantum system in terms of the inverse of a continuous, two-time quantum Fisher-information matrix.

Waveform QCRB. Spectral uncertainty principle

M. Tsang, H. M. Wiseman, and C. M. Caves, PRL 106, 090401 (2011).

$$S_{\rm est}(\omega)\left(S_{\Delta q}(\omega)+\frac{\hbar^2}{4S_{\Delta f}(\omega)}\right)\geq\frac{\hbar^2}{4}$$

$$\boxed{S_{\Delta q}(\omega)=|G(\omega)|^2 S_\xi(\omega)}$$

Prior-information term

At frequencies where there is little prior information,

$$S_{\rm est}(\omega)\geq\frac{\hbar^2}{4S_{\Delta q}(\omega)}=\frac{1}{|G(\omega)|^2}\frac{\hbar^2}{4S_\xi(\omega)}=\boxed{\frac{S_\eta(\omega)}{|G(\omega)|^2}}$$

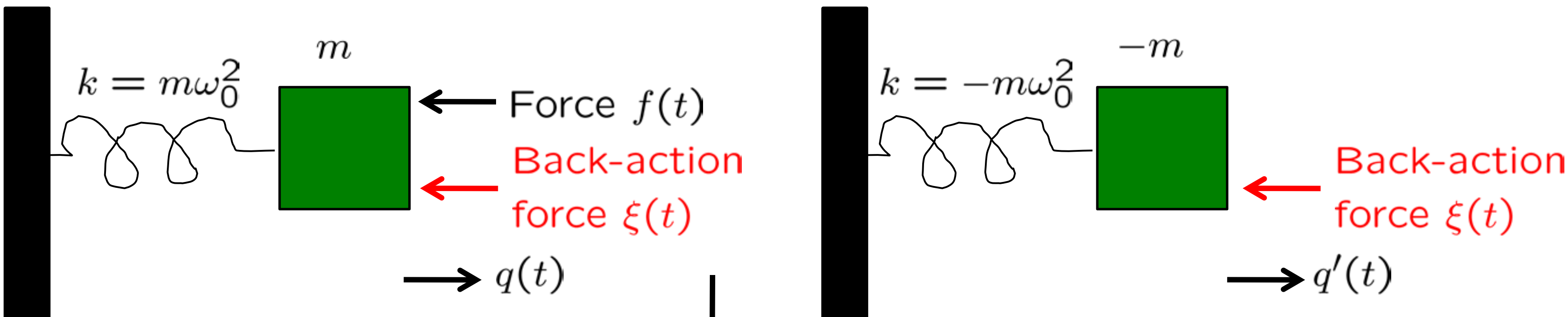
Minimum-uncertainty noise

No hint of SQL—no back-action noise, only measurement noise—but can the bound be achieved?

Beating the SQL. Strategy 3

Strategy 3. Quantum noise cancellation (QNC) using oscillator and negative-mass oscillator.

Primary oscillator — Negative-mass oscillator



Monitor collective position Q

Readout $y(t) = Q(t) + \eta(t) = q(t) + q'(t) + \eta(t)$

Measurement noise

Conjugate pairs

$$Q = q + q' \qquad P = (p + p')/2$$

$$\delta q = (q - q')/2 \qquad \delta p = p - p'$$

Oscillator pairs

QCRB

$$S_{\Delta z}(\omega) = \frac{S_\eta(\omega)}{|G(\omega)|^2}$$

Quantum noise cancellation

M. Tsang and C. M. Caves, PRL 105,123601 (2010).

Oscillator (q,p) and negative-mass oscillator (q',p')

Conjugate pairs

$$Q = q + q' \qquad P = (p + p')/2$$

$$\delta q = (q - q')/2 \qquad \delta p = p - p'$$

Oscillator pairs

Back-action noise in q and q' cancels in $Q = q + q'$

OR

$Q = q + q'$ is a new BAE observable, which, rather than being conserved, acts just like oscillator position, responding to a force in the same way.

Paired sidebands about a carrier frequency
Paired collective spins, polarized along opposite directions

W. Wasilewski , K. Jensen, H. Krauter, J. J. Renema, M. V. Balbas, and E. S. Polzik, PRL 104, 133601 (2010).

That’s it, folks! Thanks for your attention.

Echidna Gorge
Bungle Bungle Range
Western Australia