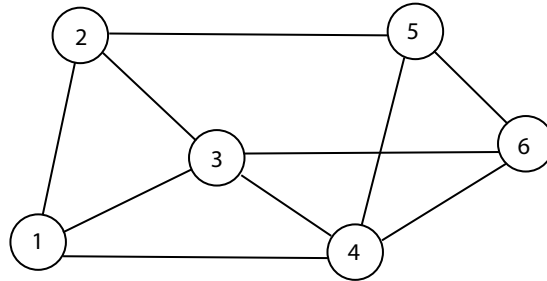# Physics 572: Homework #3
Due Date: April 18$^\text{th}$, 2019

Problem 1: Graph coloring and local Hamiltonians. Let $G = (V, E)$ be a combinatorial graph with vertex set V and edge set $E \subseteq V \times V$. If $(u, v) \in E$ for some $u, v \in V$ then we say that "u is adjacent to v in the graph G." Here is an example of a graph with 6 vertices and 10 edges,

A k-coloring of a graph is a function $f : V \to \{1, ..., k\}$ with the property

$$(u, v) \in E \implies f(u) \neq f(v), \tag{1}$$

which says that no two adjacent vertices are assigned the same color. The value $f(u)$ is called the color associated with u. For small values of k we often let the range $\{1, ..., k\}$ be a set of strings describing colors, e.g. $\{\text{red}, \text{green}, \text{blue}\}$ for a 3-coloring. We say a graph G is k-colorable if a k-coloring of G exists. The language k-COLORING is a set of descriptions of all graphs (e.g. presented as sets of vertices and edges) that are k-colorable. 3-COLORING is NP-complete.

I. (20 points) Is the example graph illustrated above 2-colorable? 3-colorable? 4-colorable? Find the minimum number of colors k necessary to color this graph, exhibit such a minimal coloring (either graphically or using the vertex labels in the figure above), and argue that $k - 1$ colors could not suffice.

II. (30 points) Give a reduction from 3-COLORING to LOCAL HAMILTONIAN. **Hint:** Let $G = (V, E)$ be an arbitrary graph. We seek to construct a local Hamiltonian which has ground energy equal to 0 if G is 3-colorable, and ground energy greater than 0 if G is not 3-colorable. For each vertex $v \in V$, one may associate a qutrit with basis states

$$|r\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad |g\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad |b\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which represent the colors red, green, and blue. **Extra hint:** As in the example of SAT, we want to write down local Hamiltonian terms that each act nontrivially on only a few qutrits and assign a non-zero energy to all the forbidden configurations (invalid colorings).

Problem 2: Circuit verifiers and the Cook-Levin theorem. Consider a Boolean circuit:
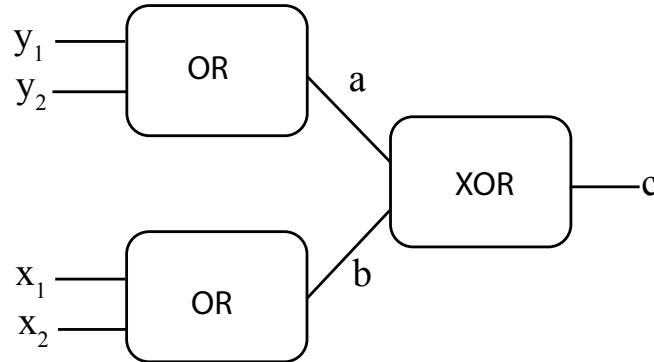
I. (10 points) Suppose this circuit belongs to a uniform family of NP verifier circuits, where the registers (x_1, x_2) are the inputs, and the registers (y_1, y_2) are the witness. Does the verifier accept the input $(x_1, x_2) = (1, 0)$?

II. (10 points) Following the proof of the Cook-Levin theorem, construct an existential Boolean formula based on this circuit (viewed as an NP verifier as described in the previous problem) that is satisfied if and only if the verifier accepts the input (x_1, x_2).

III. (20 points) Associate $x_1, x_2, y_1, y_2, a, b, c$ with qubits. Following the proof of the Cook-Levin theorem, construct a 3-local Hamiltonian H that acts on these 7 qubit in such a way that H has ground state energy equal to zero if and only if the NP verifier accepts $(x_1, x_2) = (1, 0)$.

IV. (10 points) Suppose instead we view this circuit as a BPP verifier, where this time (x_1, x_2) are the inputs, and the acceptance probability is based on the number of inputs to the register (y_1, y_2) that cause the verifier to output 1. What is the probability that the verifier outputs 1 for the input $(x_1, x_2) = (1, 0)$? Does this BPP verifier accept the input $(x_1, x_2) = (1, 0)$?

Problem 3: Amplifying measurement probabilities. In class we learned that it can be nontrivial to amplify the completeness and soundness possibilities for QMA, and saw the circuit which describes an in-place amplification protocol (using only one copy of the witness) due to Marriott and Watrous. In this problem we will analyze a simpler protocol that accomplishes a related feat: quantum bomb testing!

Our bomb factory prepares a bomb that is either a dud (it will never explode) or a live bomb. To test whether the bomb is live we need to interact with it (say, by sending a photon). But if

the bomb is live and we interact with it, then it will explode and this is undesirable. Therefore we would like some way to test whether the bomb is live, without risking an explosion[1].

To solve this problem, we will presume we can interrogate the bomb in quantum superposition. To model this, we consider two qubits: the first is a probe (e.g. a photon) and the second qubit can be thought of as a measurement device. If the bomb is live, the probe entangles with the measurement device by a CNOT, and if the bomb is a dug then the probe passes through unchanged.
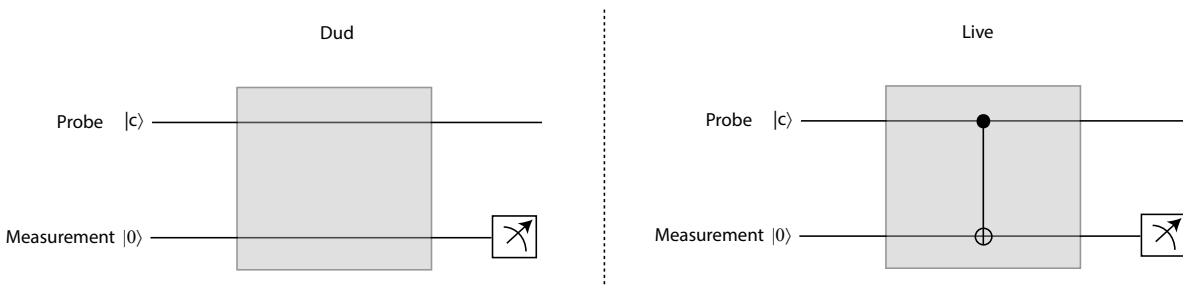
If the measurement of the second qubit ever finds the state $|1\rangle$, then the bomb explodes. Notice that if the probe is in the state $|1\rangle$ then the measurement distinguishes dud bombs from live bombs with 100% certainty. The trick to testing the bomb safely is to run the probe through the box many times while attempting to slowly rotating it from $|0\rangle$ to $|1\rangle$. Consider the rotation matrix,

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Working in the limit $\theta \ll 1$, the goal of this problem is to analyze the following protocol:
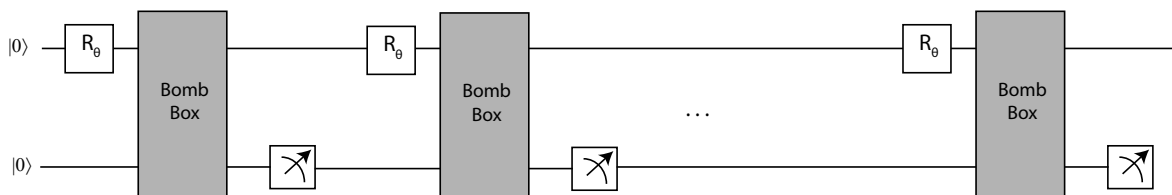
where the ellipsis indicate $\pi/(2\theta)$ repetitions of the pattern of circuit elements.

I. (20 points) If the bomb is a dud, then what is the state of the probe at the end of the protocol?

II. (30 points) If the bomb is live, then with some probability the test will fail and the bomb will explode (remember, if the measurements along the lower rail ever yield 1 then the bomb explodes). What is the probability that the test fails? If the bomb doesn't explode, then what is the state of the probe at the end of the protocol?

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[1]It is unfortunate, but standard, to use a weapons manufacturing motivation to learn this feature of quantum mechanics. As a creative writing exercise, you are encouraged to invent more pleasant alternative motivations.

Problem 4: Phase estimation. In class we learned the standard version of Kitaev's phase estimation algorithm, in which we use a register of n ancilla qubits to measure the most significant n bits of a phase $\theta \approx 0.\theta_1...\theta_n$ of an eigenvalue $\lambda = e^{2\pi i\theta}$ of a unitary operator U acting on the state $|\psi\rangle$. In addition to the n ancilla qubits, this version of phase estimation also requires an inverse quantum fourier transform on the n qubit register, which may be difficult to implement on near-term quantum computing devices.

In this problem we will study a variant of the phase estimation algorithm which only requires one ancilla qubit (instead of n ancilla qubits) and which avoids the n qubit inverse quantum Fourier transform. The tradeoff will be a need to repeat the algorithm many times to estimate the phase of the eigenvalue to high precision. Consider the following circuit:
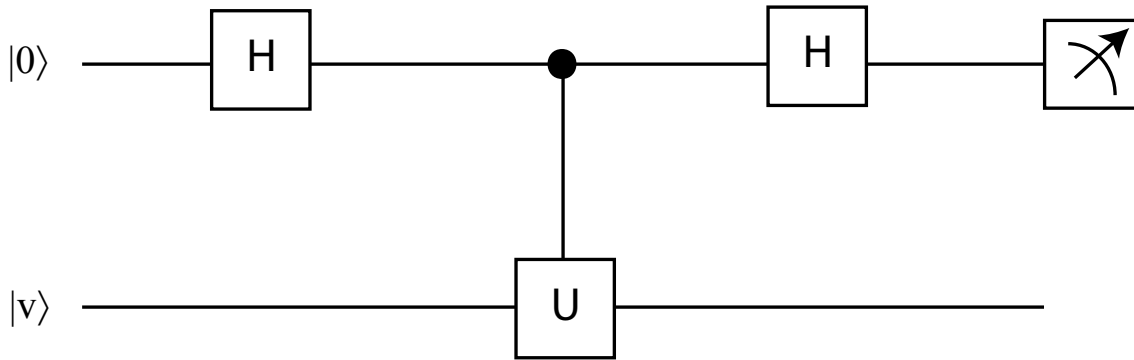
I. (20 points) Analyze the measurement probabilities for the ancilla qubit (the top line) in the circuit above, and show that they are sensitive to θ. Devise a protocol to decide whether θ is greater than $1/2$, or at most $1/2$, with a reasonably small probability of error.

II. (30 points) Building on the previous protocol, find the number of repetitions of this experiment that would suffice to estimate θ up to additive error ϵ, with a failure probability at most p_{fail}. How does the number of repetitions scale with ϵ?

Hint: Direct probabilistic reasoning is fine, or Hoeffding's inequality may be useful.