Physics 521

11/22/2000

Exam 2 Solutions

Problem 1: SHO and Classical Forces

(a) The effect of a classical force is governed by an interaction Hamiltonian $\hat{H}_{int} = -\hat{x} F_0$. This can easily be justified through the Heisenberg eqns of motion:

$$\frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \hat{H}_{int}] = -\frac{F_0}{i\hbar}\underbrace{[\hat{p}, \hat{x}]}_{= -i\hbar} = F_0$$ as expected.

(b) The impulse is sufficiently short that $\hat{H}_0$ can be neglected during time δt since $\hat{H}_0 = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$ this requires $\boxed{\omega \delta t \ll 1}$. Then no dynamical phases are accumulated from $\hat{H}_0$

Under this condition $|\psi(\delta t)\rangle = e^{-i \hat{H}_{int} \delta t/\hbar} |0\rangle$

$$\Rightarrow |\psi(\delta t)\rangle = e^{i F_0 \delta t \hat{x}/\hbar} |0\rangle = e^{i p_0 \hat{x}/\hbar} |0\rangle$$

$$= \hat{M}(p_0)|0\rangle$$

where $\hat{M}(p_0) = e^{i p_0 \hat{x}/\hbar}$ is the momentum translation operator

(c) Momentum translation is a special case of translation in phase space. To see this, express $\hat{x}$ is terms of $\hat{a}$ and $\hat{a}^{\dagger}$:

$$\hat{x} = \frac{x_c}{\sqrt{2}}(\hat{a} + \hat{a}^{\dagger}), \quad \text{where } x_c = \sqrt{\frac{\hbar}{m\omega}}$$

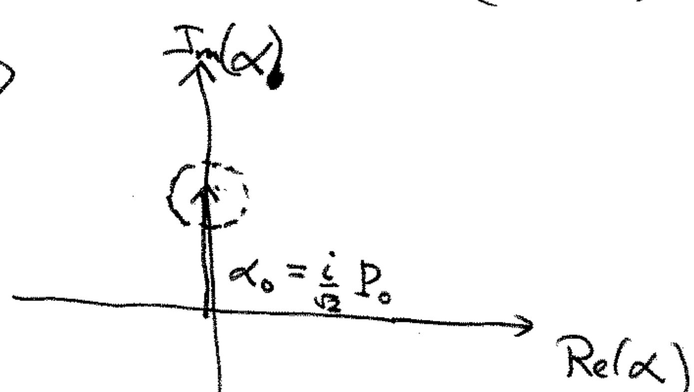
$$\Rightarrow \quad \hat{M}(p_0) = e^{\left(\frac{i p_0 x_c}{\sqrt{2}}\hat{a}^{\dagger} + \frac{i p_0 x_c}{\sqrt{2}}\hat{a}\right)}$$

$$= e^{\alpha_0 \hat{a}^{\dagger} - \alpha_0^{*}\hat{a}} = \hat{D}(\alpha_0)$$

where $\alpha_0 = \frac{i}{\sqrt{2}} P_0$, $P_0 = \frac{p_0}{p_c}$, $p_c = \sqrt{m\omega\hbar}$

Thus $\hat{M}(p_0)$ is the phase-space displacement operator to phase space point $(X_0 = 0, P_0)$

$$|\psi(\delta t)\rangle = |\alpha_0\rangle$$



Coherent state centered at $\alpha_0 = \frac{i}{\sqrt{2}} P_0$

The time evolution is then easy to find as in P.S.# , problem .

$$|\psi(t)\rangle = \hat{U}_0(t)\,|\psi(\delta t)\rangle = \hat{U}_0(t)\,|\alpha_0\rangle$$ where $\hat{U}_0 = e^{-i\hat{H}_0 t/\hbar}$

$$= \hat{U}_0(t)\,\hat{D}(\alpha_0)\,\hat{U}^\dagger(t)\,\underbrace{\hat{U}(t)\,|0\rangle}_{e^{-i\omega t/2}|0\rangle}$$

$$= e^{i\phi}\, e^{\alpha_0 \hat{U}_0(t)\hat{a}^\dagger \hat{U}_0^\dagger(t) - \alpha_0^* \hat{U}(t)\hat{a}\,\hat{U}^\dagger(t)}\,|0\rangle \qquad \left(\phi = -\frac{\omega t}{2}\right)$$

$$= e^{i\phi}\, e^{\alpha_0 e^{-i\omega t}\hat{a}^\dagger - \alpha_0^* e^{i\omega t}\hat{a}}\,|0\rangle$$

$$= e^{i\phi}\,\hat{D}(\alpha(t))\,|0\rangle = e^{i\phi}\,|\alpha(t)\rangle$$

where $\alpha(t) = \alpha_0 e^{-i\omega t} = \frac{1}{\sqrt{2}}\, i P_0 e^{-i\omega t}$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{P_0 \sin\omega t}_{= X_{class}(t)} + i\,\underbrace{P_0 \cos\omega t}_{P_{class}(t)}\Big)$$

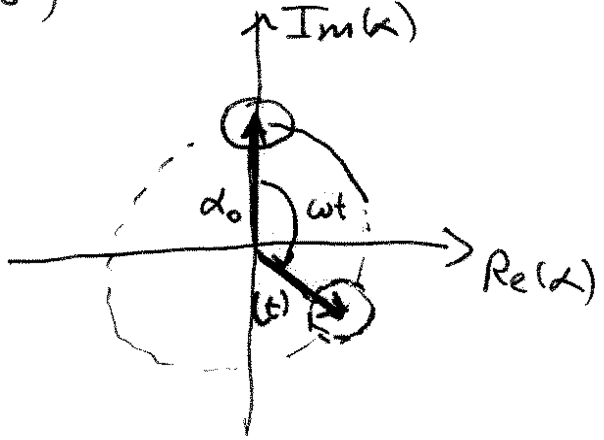
$$x_{class}(t) = x_c\, X_{class} \qquad p_{class} = p_c\, P_{class}$$

<u>with units</u>

The classical trajectory with initial condition

$x_{class} = \frac{P_0}{m\omega}\sin\omega t \qquad p(t) = p_0 \cos\omega t \qquad x(0) = 0, \quad p(0) = p_0$

$\left(P_0 = p_c P_0\right)$



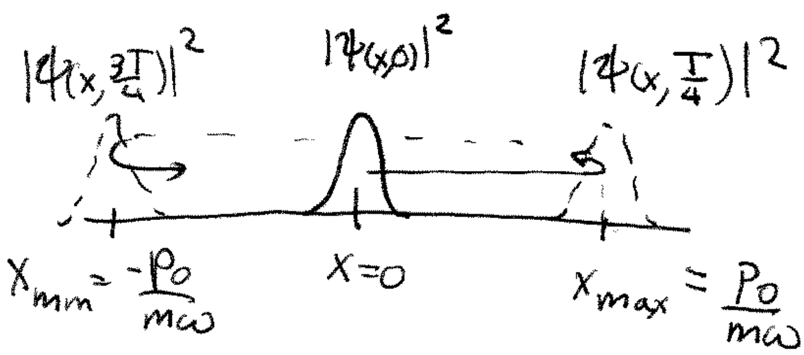
Phase space trajectory

The wave function in position space

$$\psi(x,t) = e^{i\phi}\langle x|\alpha(t)\rangle = e^{i\phi}\, e^{ip}\; u_0(x - x_{class}(t))$$

$$\boxed{\psi(x,t) = e^{i\phi}\, e^{ip_0 \sin\omega t\, x/\hbar}\, \frac{1}{(\sqrt{\pi}\, x_c)^{1/2}}\, e^{-(x - \frac{p_0}{m\omega}\sin\omega t)^2/2x_c^2}}$$

Gaussian wave packet whose position dependent phase varies in time according to $p_{class}(t)$ and whose center oscillates along $x_{class}(t)$



(d) We have $|\psi(t)\rangle = e^{i\phi}\,|\alpha(t)\rangle$

$$\langle E\rangle = \langle\psi(t)|\hat{H}_0|\psi(t)\rangle = \langle\ \ |\hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})|\alpha(t)\rangle$$

$$= \hbar\omega\left[|\alpha(t)|^2 + \tfrac{1}{2}\right]$$ since $\hat{a}|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$, $\hat{a}^\dagger|\alpha(t)\rangle = \alpha^*(t)\,|\alpha(t)\rangle$

$$= \hbar\omega\left[|\alpha_0|^2 + \tfrac{1}{2}\right]$$: independ of time as expected

Plug in $\alpha_0 = \frac{i}{\sqrt{2}}P_0 = \frac{i\,p_0}{\sqrt{2m\hbar\omega}}$ $\Rightarrow$ $\boxed{\langle E\rangle = \frac{p_0^2}{2m} + \frac{\hbar\omega}{2}}$

The average energy is the zero-point plus the "recoil kinetic energy" of the impulse $\frac{p_0^2}{2m}$

$$\Delta E^2 = \langle\hat{H}_0^2\rangle - \langle\hat{H}_0\rangle^2$$

$$\langle\hat{H}_0^2\rangle = \langle\psi(t)|(\hbar\omega)^2(\hat{N}+\tfrac{1}{2})^2|\psi(t)\rangle = (\hbar\omega)^2\langle\psi(t)|\hat{N}^2+\hat{N}|\psi(t)\rangle + \frac{(\hbar\omega)^2}{4}$$

Aside: $\langle\psi(t)|\hat{N}^2|\psi(t)\rangle = \langle\psi(t)|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\psi(t)\rangle$

(use commutator) $= \langle\psi(t)|(\hat{a}^\dagger)^2(\hat{a})^2|\psi(t)\rangle + \langle\psi(t)|\hat{a}^\dagger\hat{a}|\psi(t)\rangle$

$$= (\alpha^*(t))^2(\alpha(t))^2 + \alpha^*(t)\alpha(t)$$

$$= |\alpha_0|^4 + |\alpha_0|^2$$

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thus $\langle \hat{H}_0^2 \rangle = (\hbar\omega)\left(|\alpha_0|^4 + 2|\alpha_0|^2 + \frac{1}{4}\right)$

$$= \left[\hbar\omega\left(|\alpha_0|^2 + \tfrac{1}{2}\right)\right]^2 + (\hbar\omega)\,|\alpha_0|^2$$

$$= \langle \hat{H}_0 \rangle^2 + \hbar\omega\left(\frac{P_0^2}{2m}\right)$$

$\Rightarrow$ $\boxed{\Delta E = \sqrt{\hbar\omega \frac{P_0^2}{2m}}}$ time independent, again as expected

To be in a truly classical regime we must insist

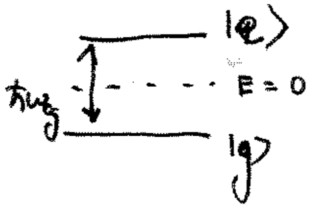
$\frac{\Delta E}{\langle E \rangle} << 1 \Rightarrow \sqrt{\frac{\hbar\omega}{P_0^2/2m}} << 1$ if $\frac{\hbar\omega}{2} << \frac{P_0^2}{2m}$

$\Rightarrow$ Require $\boxed{\frac{P_0^2}{2m} >> \hbar\omega}$

this makes sense physically. If $\frac{P_0^2}{2m} << \hbar\omega$ the impulse energy is small compared to the energy gap between the ground and first excited state. However for a large impulse, we obtain a highly excited packet which is a superposition of many energy levels, but with a dispersion in number much smaller than $\langle \hat{N} \rangle$.

Problem 2: Pseudo-spin picture of atom photon interaction

Two level atom



$$\hat{H}_A = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z$$

where $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$

i.e. $\left(|e\rangle \equiv |+_z\rangle \quad |g\rangle \equiv |-_z\rangle\right)$

(a) Given dipole-dipole Hamiltonian:

$$\hat{H} = \frac{\hbar\omega_{eg}}{2}\left(\hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)}\right) + V_{dd}\left(\hat{\sigma}_+^{(1)} \otimes \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_+^{(2)}\right)$$

Let us find the matrix representation of $\hat{H}$ in the "uncoupled basis" $\{|ee\rangle, |e,g\rangle, |g,e\rangle, |g,g\rangle\}$

$$\hat{H}|e,e\rangle = \frac{\hbar\omega_{eg}}{2}(1+1)|e,e\rangle + 0 \quad \text{since } \hat{\sigma}_+|e\rangle = 0$$

$$= \hbar\omega_{eg}|ee\rangle$$

$$\hat{H}|e,g\rangle = \frac{\hbar\omega_{eg}}{2}(1-1)|e,g\rangle + V_{dd}(0 + |g,e\rangle) =$$

$$= V_{dd}|g,e\rangle$$

$$\hat{H}|g,e\rangle = \frac{\hbar\omega_{eg}}{2}(-1+1)|g,e\rangle + V_{dd}(|e,g\rangle + 0) =$$

$$= V_{dd}|e,g\rangle$$

$$\hat{H}|g,g\rangle = \frac{\hbar\omega_{eg}}{2}(-1-1)|g,g\rangle + 0$$

$$= -\hbar\omega_{eg}|g,g\rangle$$

Thus the only nonvanishing matrix elements are:

$$\langle ee|\hat{H}|ee\rangle = \hbar\omega_{eg} \qquad \langle gg|\hat{H}|gg\rangle = -\hbar\omega_{eg}$$

$$\langle eg|\hat{H}|ge\rangle = \langle ge|\hat{H}|eg\rangle = V_{dd}$$

$$\Rightarrow \hat{H} \doteq \begin{array}{c} \left[\begin{array}{c|cc|c} \hbar\omega_{eg} & 0 & 0 & 0 \\ \hline 0 & 0 & V_{dd} & 0 \\ 0 & V_{dd} & 0 & 0 \\ \hline 0 & 0 & 0 & -\hbar\omega_{eg} \end{array}\right] \end{array} \begin{array}{l} |ee\rangle \\ |eg\rangle \\ |ge\rangle \\ |gg\rangle \end{array}$$

$\hat{H}$ is thus "Block-diagonal", with only $|eg\rangle$ and $|ge\rangle$ coupled.
We must thus diagonalize in the 2D subspace spanned by $\{|eg\rangle, |ge\rangle\}$

$$\hat{H} \doteq V_{dd}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{array}{l} |eg\rangle \\ |ge\rangle \end{array}$$

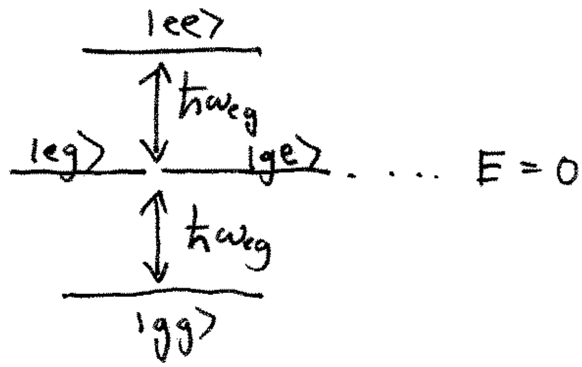
We have seen this matrix before.
It's eigenvalues are $\pm V_{dd}$ with eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$

Thus, the stationary states are:

$$\boxed{\begin{array}{l} |ee\rangle : \text{ Eigenvalue } \hbar\omega_{eg} \\ |gg\rangle : \text{ Eigenvalue } -\hbar\omega_{eg} \\ \frac{1}{\sqrt{2}}(|eg\rangle \pm |ge\rangle) \equiv |\psi_\pm\rangle : \text{ Eigenvalues } \pm V_{dd} \end{array}}$$

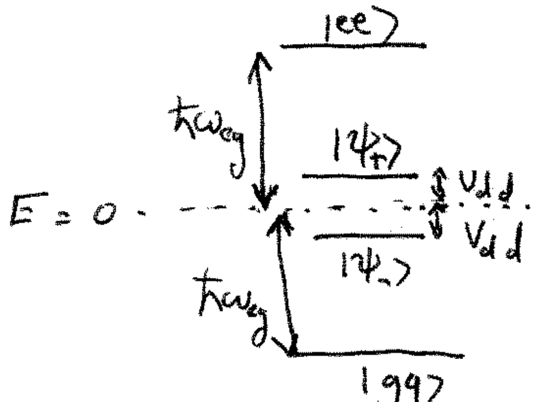
Energy level diagram:

Note: If $V_{dd} = 0$, the energy level diagram for the two atoms is



The states $|eg\rangle$ and $|ge\rangle$ are degenerate.

When the coupling is turned on, the degeneracy is broken



Assuming $V_{dd} > 0$ and $V_{dd} < \hbar\omega_{eg}$

The eigenstates are the "coupled representation" of total spin

"Triplet"

$|J=1, M=1\rangle = |ee\rangle$

$|J= \ , M=0\rangle = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)$
$\equiv |\psi_+\rangle$

$|J=1, M=-1\rangle = |gg\rangle$

"Singlet"

$|J=0, M=0\rangle = \frac{1}{\sqrt{2}}(|eg\rangle - |ge\rangle)$
$= |\psi_-\rangle$

(b) The coupling of an atom to a monochromatic laser field is described by the Hamiltonian: $\hat{H}_{AL} = -\frac{\hbar\omega}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$

$$= -\hbar\omega\,\hat{S}_z + \hbar\Omega\,\hat{S}_x$$

For two atoms $\hat{H}_{int} = -\hbar\omega\,\hat{J}_z + \hbar\Omega\,\hat{J}_x = \hat{H}_{A_1 L} + \hat{H}_{A_2 L}$

Where $\hat{J}_z = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$ $\quad \hat{J}_x = \hat{S}_x^{(1)} + \hat{S}_x^{(2)}$ are the total angular momentum operators

The coupled basis states are eigenstates of $\hat{J}^2$ and $\hat{J}_z$

Triplet: $|J=1, M=1\rangle = |e\rangle|e\rangle$

$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle|g\rangle + |g\rangle|e\rangle\right) = |\Psi_+\rangle$

$|J=1, M=-1\rangle = |g\rangle|g\rangle$

Singlet: $|J=0, M=0\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle|g\rangle - |g\rangle|e\rangle\right) = |\Psi_-\rangle$

The operators $\hat{J}_x$ and $\hat{J}_z$ <u>do not</u> connect spaces with different total angular momentum quantum # = J.

We found matrix representations of $\hat{J}_x$ and $\hat{J}_z$ for $J=1$

$$\hat{J}_z \doteq \begin{array}{cc} & M \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{matrix} 1 \\ 0 \\ -1 \end{matrix} \end{array} \qquad \hat{J}_x \doteq \frac{1}{\sqrt{2}} \begin{array}{cc} & M \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 0 \\ -1 \end{matrix} \end{array}$$

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Of course in $j=0$ $\vec{j} \doteq [0]$

Thus we have a representation of $\hat{H}_{A_1L} + \hat{H}_{A_2L}$ in the coupled basis:

$$\hat{H}_{int} = \left[\begin{array}{ccc|c} -\hbar\omega & \frac{\hbar\Omega}{\sqrt{2}} & 0 & 0 \\ \frac{\hbar\Omega}{\sqrt{2}} & 0 & \frac{\hbar\Omega}{\sqrt{2}} & 0 \\ 0 & \frac{\hbar\Omega}{\sqrt{2}} & \hbar\omega & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} |J=1, M=1\rangle = |ee\rangle \\ |J=1, M=0\rangle = |\Psi_+\rangle \\ |J=1, M=-1\rangle = |gg\rangle \\ |J=0, M=0\rangle = |\Psi_-\rangle \end{array}$$

Note that the singlet state $|\Psi_-\rangle$ has no non-zero matrix elements with any state; i.e. it is not coupled by the laser field to any other state. Such a state is know as a "dark state" in laser spectroscopy.

The method we used here was based on the theory of addition of angular momentum. A more brute force approach is to use the method in part (a).

(c) Given N 2-level atoms.

$$|g\rangle|g\rangle|g\rangle \cdots |g\rangle = |J = N/2 ; M = -N/2\rangle$$

Case $N=3$: $\boxed{|J=3/2, M=-3/2\rangle = |g\rangle|g\rangle|g\rangle}$

To find other M states, apply rasing operator

Remember $\hat{J}_+ |J, M\rangle = \sqrt{J(J+1) - M(M+1)}\, |J, M+1\rangle$

Here $\hat{J}_+ = \hat{\sigma}_+^{(1)} \otimes \hat{\mathbb{1}}^{(2)} \otimes \hat{\mathbb{1}}^{(3)} + \hat{\mathbb{1}}^{(1)} \otimes \hat{\sigma}_+^{(2)} \otimes \hat{\mathbb{1}}^{(3)} + \hat{\mathbb{1}}^{(1)} \otimes \hat{\mathbb{1}}^{(2)} \otimes \hat{\sigma}_+^{(3)}$

$$\Rightarrow \hat{J}_+ |J=3/2, M=-3/2\rangle = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) + \tfrac{3}{2}(-\tfrac{3}{2}+1)}\, |J=3/2, M=-1/2\rangle$$

$$= \sqrt{3}\, |J=3/2, M=-1/2\rangle = |e\rangle|g\rangle|g\rangle + |g\rangle|e\rangle|g\rangle + |g\rangle|g\rangle|e\rangle$$

$$\Rightarrow \boxed{|J=3/2, M=-1/2\rangle = \frac{1}{\sqrt{3}}\left(|e\rangle|g\rangle|g\rangle + |g\rangle|e\rangle|g\rangle + |g\rangle|g\rangle|e\rangle\right)}$$

$$J_+ |J=3/2, M=-1/2\rangle = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(-\tfrac{1}{2}+1)}\, |J=3/2, M=+\tfrac{1}{2}\rangle$$

$$= 2\, |J=3/2, M=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\Big(\cancel{|e\rangle|g\rangle|g\rangle} + |e\rangle|e\rangle|g\rangle + |e\rangle|g\rangle|e\rangle +$$
$$+ |e\rangle|e\rangle|g\rangle + |g\rangle|e\rangle|e\rangle +$$
$$+ |e\rangle|g\rangle|e\rangle + |g\rangle|e\rangle|e\rangle \Big)$$

$$\Rightarrow \boxed{|J=\tfrac{3}{2}, M=+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|e\rangle|e\rangle|g\rangle + |e\rangle|g\rangle|e\rangle + |g\rangle|e\rangle|e\rangle\right)}$$

Finally: $\hat{J}_+ |J=3/2, M=1/2\rangle = \sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}\ |J=3/2, M=3/2\rangle$

$$= \sqrt{3}\ |J=3/2, M=3/2\rangle = \frac{1}{\sqrt{3}}\left(3\ |e\rangle|e\rangle|e\rangle\right)$$

$$\Rightarrow \boxed{|J=3/2, M=3/2\rangle = |e\rangle|e\rangle|e\rangle}$$

(d) Extra credit. Suppose we start with all atoms in the ground state and then couple them with a laser. What states can we reach?

The total Hamiltonian is:

$$\hat{H} = \sum_{i=1}^{N}\left(\hat{H}_{A_i} + \hat{H}_{A_i L}\right) = -\hbar\Delta\hat{J}_z + \hbar\Omega\hat{J}_x$$

where $\Delta \equiv \omega - \omega_{eg}$

$$\hat{J}_z = \sum_i \hat{S}_z^{(i)} \qquad \hat{J}_x = \sum_i \hat{S}_x^{(i)}$$

$\Rightarrow$ State at a later time

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\ |\psi(0)\rangle = e^{-i(\Omega t\,\hat{J}_x - \Delta t\,\hat{J}_z)}\ |\psi(0)\rangle$$

$$= e^{-i\theta(t)(\vec{n}(t)\cdot\hat{\vec{J}})}\ |\psi(0)\rangle$$

where $\theta(t) = \sqrt{(\Omega t)^2 + (\Delta t)^2} = \left(\sqrt{\Omega^2+\Delta^2}\right)t$

and $\theta(t)\,\vec{n}(t) = \Omega t\ \vec{e}_x - \Delta t\ \vec{e}_z$

Thus the time translation operator is a <u>rotation operator</u>: $|\psi(t)\rangle = \hat{D}_{\vec{n}(t)}(\theta(t))\, |\psi(0)\rangle$

The effect of a rotation cannot change the <u>total</u> <u>angular momentum</u>. Thus, since we start in the state $|J=\frac{N}{2}, M=-\frac{N}{2}\rangle$, all other states connect through the laser field must have $J=\frac{N}{2}$. All other states are "dark states". We saw an example of this in (b) for $N=2$. Only state with $J=\frac{N}{2}=1$ are connected by the laser. The remaining state with $J=0$ is "dark".