Physics 522. Quantum Mechanics II

Problem Set #3 – Tensors and the Wigner-Eckart Theorem

Due Monday, Mar. 5, 2001

Problem 1: Light-shift for multilevel atoms (15 points)

In P.S. #1, we found the AC-Stark (light shift) for the case of a two-level atom driven by a monochromatic field. The ground state energy level shift was $\Delta E_g = -\alpha |E_0|^2 / 4$, where $\alpha = -\left|\langle e|\hat{d}|g\rangle\right|^2 / (\hbar\Delta)$ is the atomic "polarizability" with $\Delta = \omega_L - \omega_{eg}$ the detuning of the applied "laser frequency" from the atomic resonance frequency. In this problem we want to look at this phenomenon in a more general context, including arbitrary polarization of the electric field, and atoms with multiple sublevels.

Consider then a general monochromatic electric field $\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\left(\mathbf{E}(\mathbf{x})e^{-i\omega_L t}\right)$, driving an atom near resonance on the transition, $|g;J_g\rangle \rightarrow |e;J_e\rangle$, where the ground and excited manifolds are each described by some total angular momentum J with degeneracy $2J$+1. The generalization of the AC-Stark shift is now the light-shift operator acting on the $2J_g+1$ dimensional ground manifold:

$$\hat{V}_{LS}(\mathbf{x}) = -\frac{1}{4}\mathbf{E}^*(\mathbf{x}) \cdot \hat{\overleftrightarrow{\alpha}} \cdot \mathbf{E}(\mathbf{x}).$$

Here $\hat{\overleftrightarrow{\alpha}} = -\dfrac{\hat{\mathbf{d}}_{ge}\hat{\mathbf{d}}_{eg}}{\hbar\Delta}$ is the atomic polarizability tensor operator, where $\hat{\mathbf{d}}_{eg} \equiv \hat{P}_e\,\hat{\mathbf{d}}\,\hat{P}_g$ is the dipole operator, projected between the ground and excited manifolds; the projector onto the excited manifold is, $\hat{P}_e = \sum_{M_e=-J_e}^{J_e} |e;J_e,M_e\rangle\langle e;J_e,M_e|$, and similarly for the ground.

(a) By expanding the dipole operator in the spherical basis, **show** that the polarizability operator can be written,

$$\hat{\overleftrightarrow{\alpha}} = \tilde{\alpha}\left(\sum_{q,M_g}\left|C_{M_g}^{M_g+q}\right|^2 \vec{\mathbf{e}}_q |g;J_g,M_g\rangle\langle g;J_g,M_g|\vec{\mathbf{e}}_q^* + \sum_{q\neq q',M_g} C_{M_g+q-q'}^{M_g+q}\,C_{M_g}^{M_g+q}\ \vec{\mathbf{e}}_{q'}|g;J_g,M_g+q-q'\rangle\langle g;J_g,M_g|\vec{\mathbf{e}}_q^*\right),$$

$$\text{where } \tilde{\alpha} \equiv -\frac{\left|\langle e;J_e\|d\|g;J_g\rangle\right|^2}{\hbar\Delta} \text{ and } C_{M_g}^{M_e} \equiv \langle J_e\,M_e|1q\,J_g\,M_g\rangle.$$

Explain physically, using dipole selection rules, the meaning of the expression for $\hat{\overleftrightarrow{\alpha}}$.

(b) Consider a polarized plane wave, with complex amplitude of the form, $\mathbf{E}(\mathbf{x}) = E_1\vec{\varepsilon}_L\,e^{i\mathbf{k}\cdot\mathbf{x}}$ where E_1 is the amplitude and $\vec{\varepsilon}_L$ the polarization (possibly complex). For an atom driven on the transition $|g;J_g=1\rangle \rightarrow |e;J_e=2\rangle$ and the cases (i) linear polarization along z, (ii) positive helicity polarization, (iii) linear polarization along x, **find** the eigenvalues and eigenvectors of the light-shift operator. Express the eigenvalues in units of $V_1 = -\frac{1}{4}\tilde{\alpha}|E_1|^2$. Please ***comment*** on what you find for cases (i) and (iii). **Repeat** for $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$ and **comment**.

(c) A deeper insight into the light-shift potential can be seen by expressing the polarizability operator in terms of irreducible tensors. **Show** that we get the sum of scalar, vector, and rank-2 irreducible tensor interaction,

$$\hat{V}_{LS} = -\frac{1}{4}\left(|\mathbf{E}(\mathbf{x})|^2 \hat{\alpha}^{(0)} + (\mathbf{E}^*(\mathbf{x}) \times \mathbf{E}(\mathbf{x})) \cdot \hat{\alpha}^{(1)} + \mathbf{E}^*(\mathbf{x}) \cdot \hat{\alpha}^{(2)} \cdot \mathbf{E}(\mathbf{x})\right),$$

$$\text{where } \hat{\alpha}^{(0)} = \frac{\hat{\mathbf{d}}_{ge} \cdot \hat{\mathbf{d}}_{eg}}{-3\hbar\Delta},\ \hat{\alpha}^{(1)} = \frac{\hat{\mathbf{d}}_{ge} \times \hat{\mathbf{d}}_{eg}}{-2\hbar\Delta},\ \hat{\alpha}_{ij}^{(2)} = \frac{1}{-\hbar\Delta}\left((\hat{\mathbf{d}}^i_{ge}\hat{\mathbf{d}}^j_{ge} + \hat{\mathbf{d}}^i_{ge}\hat{\mathbf{d}}^j_{ge})/2 - \hat{\alpha}^{(0)}\delta_{ij}\right).$$

(d) For the particular case of $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$, **show** that the rank-2 tensor part *vanishes*. **Show** that the light-shift operator can be written in a basis independent *form* of a scalar interaction (independent of the sublevel), plus an effective Zeeman interaction for a fictitious B-field interacting with the spin 1/2 ground state,

$$\hat{V}_{LS} = V_0(\mathbf{x})\hat{1} + \mathbf{B}_{fict}(\mathbf{x}) \cdot \hat{\vec{\sigma}}$$

<u>Extra credit for the following three parts (10 points)</u>

(e) Explicitly show that

$$V_0(\mathbf{x}) = \frac{2}{3}U_1|\vec{\varepsilon}_L(\mathbf{x})|^2 \text{ (proportional to field intensity) and}$$

$$\mathbf{B}_{fict}(\mathbf{x}) = \frac{1}{3}U_1\left(\frac{\vec{\varepsilon}_L^*(\mathbf{x}) \times \vec{\varepsilon}_L(\mathbf{x})}{i}\right), \text{ (proportional to the field ellipticity),}$$

and I have written $E(\mathbf{x}) = E_1\vec{\varepsilon}_L(\mathbf{x})$. Use this form to **explain** your results form part (b) on the transition $|g; J_g = 1/2\rangle \rightarrow |e; J_e = 3/2\rangle$.

Consider now the non-plane wave case. Suppose we have a superposition of plane waves, e.g. two counterpropagating cross-polarized linearly plane waves (this configuration is know as lin⊥lin, and plays an important role in the theory of laser cooling). The spatial amplitude is,

$$\mathbf{E}(z) = E_1 e^{ikz}\,\vec{\mathbf{e}}_x + E_1 e^{-ikz}\,\vec{\mathbf{e}}_y = \sqrt{2}V_1\left(e^{-i\pi/4}\sin(kz - \frac{\pi}{4})\vec{\mathbf{e}}_+ + e^{i\pi/4}\cos(kz - \frac{\pi}{4})\vec{\mathbf{e}}_-\right).$$

The second expression shows that the field can be viewed as a superposition of right and left helicity standing waves, with the nodes of one matching the antinodes of the other.

(f) In lin⊥lin **show** that the light-shift operator is invariant under a rotation by 180° about the z-axis and thus the eigenvectors must be superpositions of only even or odd M_g.

(g) In lin⊥lin, **show** that the light shift operator is

$$\hat{V}_{LS}(z) = 2V_1 \begin{pmatrix} \sum_{M_g} \left(\left| C_{M_g}^{M_g+1} \right|^2 \sin^2(kz - \pi/4) + \left| C_{M_g}^{M_g-1} \right|^2 \cos^2(kz - \pi/4) \right) \left| g; J_g, M_g \right\rangle \left\langle g; J_g, M_g \right| \\ - \frac{i}{2} \sum_{M_g} C_{M_g+2}^{M_g+1} C_{M_g}^{M_g+1} \sin(2kz - \pi/2) \left| g; J_g, M_g + 2 \right\rangle \left\langle g; J_g, M_g \right| + h.c \end{pmatrix}$$

The diagonal terms in the magnetic sublevels are known as the "diabatic potentials". The eigenvalues as a function of *z* including the off-diagonal terms, are known as the "adiabatic potentials", since they represent the internal eigenvalues if the atom where to adiabatically move along the axis. **Plot** the diabatic and adiabatic potentials for the two cases: $\left| g; J_g = 1/2 \right\rangle \rightarrow \left| e; J_e = 3/2 \right\rangle$ and $\left| g; J_g = 1 \right\rangle \rightarrow \left| e; J_e = 2 \right\rangle$. Are all crossings between the diabatic potentials removed in the diagonalization? If not why.

Problem 2: Landé Projection Theorem (10 Points)

The Landé Projection Theorem (LPT) is a special case of the Wigner-Eckart Theorem for the case that the initial and final state are the same angular momentum. It states:

$$\langle \alpha'; j\, m' | \hat{\mathbf{V}} | \alpha; jm \rangle = \frac{\langle \alpha'; j \,| \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j \rangle}{j(j+1)} \langle j\, m' | \hat{\mathbf{J}} | \, jm \rangle, \text{ where } \hat{\mathbf{V}} \text{ is a vector operator w.r.t. } \hat{\mathbf{J}}.$$

(a) Give a geometric interpretation of this in terms of a vector picture.

(b) To prove this theorem, take the following steps (do not give verbatim, Sakurai's derivation):

(i) Show that $\langle \alpha'; j, m \,| \hat{\mathbf{J}} \cdot \hat{\mathbf{V}} | \alpha; j, m \rangle = \langle \alpha'; j \| J \| \alpha'; j \rangle \langle \alpha'; j \| V \| \alpha; j \rangle$, *independent of* m.

(ii) Use this to show, $\left| \langle \alpha; j \| J \| \alpha; j \rangle \right|^2 = j(j+1)$ *independent of* α.

(iii) Show that $\langle j\, m' | 1 q\, j m \rangle = \langle j m' | \hat{J}_q | \, jm \rangle / \sqrt{j(j+1)}$.

(iv) Put it all together to prove the LPT.

(c) As an application of the LPT, consider the Zeeman interaction for hydrogen.

$$\hat{H}_{\text{int}} = -\hat{\vec{\mu}} \cdot \mathbf{B},$$

where the magnetic dipole operators is $\hat{\vec{\mu}} = -\mu_B (g_l \hat{\mathbf{L}} + g_s \hat{\mathbf{S}})$, with $g_l = 1$, $g_s = 2$.

If the Zeeman effect is small compared to the fine-structure, but ignoring hyperfine structure, use the LPT to show that in a state nL_J, the magnetic moment has the form,

$$\hat{\vec{\mu}} = -g_J \mu_B \hat{\vec{\mathbf{J}}}, \text{ where } g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \text{ is known as the Landé g-factor.}$$

Hint: Prove then use $\mathbf{J}\cdot\mathbf{L} = \mathbf{L}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)$ and $\mathbf{J}\cdot\mathbf{S} = \mathbf{S}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)$.

(d) Estimate the magnetic field strength at which the Zeeman interaction becomes on the order of the fine structure splitting between the $2p_{1/2}$ and $2p_{3/2}$ state in hydrogen.

Problem 3: Natural lifetimes of Hydrogen (10 points)

Though in the absence of any perturbation, an atom in the excited state will stay there forever (it is a stationary state), in reality it will "spontaneously decay" to the ground state. Fundamentally this occurs because the atom is always perturbed by 'vacuum fluctuations" in the electro-magnetic field. We will find later in the semester, the spontaneous emission rate on a dipole allowed transition from initial excited state $|\psi_e\rangle$ to all allowed grounds states $|\psi_g\rangle$ is ,

$$\Gamma = \frac{4}{3\hbar}k^3 \sum_g \left|\langle\psi_g|\hat{\mathbf{d}}|\psi_e\rangle\right|^2, \text{ where } k = \omega_{eg}/c \text{ is the emitted photon's wave number.}$$

Consider now hydrogen including fine-structure. For a given sublevel, the sponteneous emission rate is

$$\Gamma_{(nLJM_J)\to(n'L'J')} = \frac{4}{3\hbar}k^3 \sum_{M_J'} \left|\langle n'L'J'M_J'|\hat{\mathbf{d}}|nLJM_J\rangle\right|^2 .$$

(a) Show that the spontaneous emission rate is *independent* of the initial M_J. Explain this result physically.

(b) Calculate the lifetime ($\tau=1/\Gamma$) of the $2p_{1/2}$ state in seconds.