Causal Nonseparability in Quantum Mechanics

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What Even Is Causality?

Without too much detail: influence on "future" events by "past" events.

Probability-theoretically, say A **causally influences** B when $\partial_A P(B|A) \neq 0$. If $\forall_A \partial_A P(B|A) = 0$ then B is not influenced by A, or not "after" A.

Causal relations in classical and relativistic physics are one-way (aside from CTCs) and subluminal, defined by arrow of time and determinism of dynamics.

What about quantum mechanics? Set aside Hamiltonian evolution (which is time-reversal symmetric and so causally trivial), consider measurements, as Časlav Brukner and other have.[1]

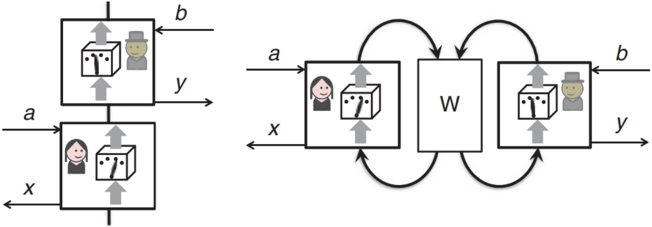
[1] Č. Brukner. "Quantum causality". In: *Nature Phys.* 10 (2014), pp. 259–263.

Scenario

For most basic case, consider Alice and Bob in two isolated labs some(where/when) in spacetime.[2]

Alice receives known state $\rho_{A1} \in \mathcal{H}_{A1}$, transforms and measures $\mathcal{M}^A_{a,x}(\rho_{A1}) = P(x|a)\rho_{A2}$ with input (operation choice) a and output (measurement outcome) x for CP trace non-increasing map $\mathcal{M}^A_{a,x} : \mathcal{H}_{A1} \to \mathcal{H}_{A2}$, and sends it out of lab $\rho_{A2} \in \mathcal{H}_{A2}$.

Similarly, Bob receives $\rho_{B1} \in \mathcal{H}_{B1}$, performs $\mathcal{M}^B_{b,y} : \mathcal{H}_{B1} \to \mathcal{H}_{B2}$, and sends $\rho_{B2} \in \mathcal{H}_{B2}$.



[2] Č. Brukner, F. Costa, and O. Oreshkov. "Quantum correlations with no causal order". In: *Nat. Commun.* 3 (2012), p. 1092.

Signalling and Causal Structure

The total probability of outcomes x, y given operations a, b is $P(x, y|a, b)$. Bob's outcomes unaffected by Alice's decisions if $\sum_x P(x, y|a, b) = P(y|b)$ independent of a, in which case Alice can't signal to Bob, or $A \not\preceq B$.

$P_{A \not\preceq B}(x, y|a, b)$ is consistent with direct causal order $B \to A$ or separated labs. Separable causal orders produce classical probability distributions: $P(x, y|a, b) = \lambda P_{A \not\preceq B} + (1 - \lambda) P_{B \not\preceq A}$ for $\lambda \in [0, 1]$, even if ignorant of causal direction.[3]

[3] Brukner, Costa, and Oreshkov, "Quantum correlations with no causal order".

Motivations

- Rebuilding QM without prior time determination allows compatibility with GR $\rightarrow$ quantum gravity?
- If it's mathematically expressible, can general causal orders be found in nature? If so, where? If not, *why?*
- New causal structures might provide technological or computational benefit.

Process Matrices

For general spacetime relation between $\mathcal{H}_{A1}$, $\mathcal{H}_{A2}$, $\mathcal{H}_{B1}$, and $\mathcal{H}_{B2}$, transform CPTP maps to positive-semidefinite by Choi-Jamiołkowski (CJ) isomorphism $M_{a,x}^{A_1 A_2} \equiv \mathbb{I} \otimes \mathcal{M}_{a,x}^{A}(|\phi_+\rangle\langle\phi_+|) \in \mathcal{H}_{A1} \otimes \mathcal{H}_{A2}$.[a]

Then $P(x, y|a, b) = \mathrm{Tr}\Big(W^{A_1 A_2 B_1 B_2}(M_{a,x}^{A_1 A_2} \otimes M_{b,y}^{B_1 B_2})\Big)$ for some **process matrix** $W^{A_1 A_2 B_1 B_2} \in \mathcal{H}_{A1} \otimes \mathcal{H}_{A2} \otimes \mathcal{H}_{B1} \otimes \mathcal{H}_{B2}$.[b]

[a] Brukner, Costa, and Oreshkov, "Quantum correlations with no causal order".

[b] Brukner, "Quantum causality".

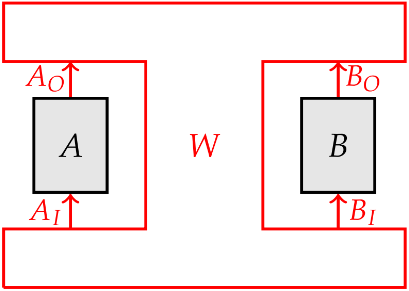


Figure: Araújo et al., "Witnessing causal nonseparability"

(Non)separability

Just like probability distributions, The process matrix is constrained to result in nonnegative probabilities that sum to 1. A process matrix W_{sep} is **causally separable** if $W_{sep} = \lambda W_{A \not\preceq B} + (1-\lambda) W_{B \not\preceq A}$ for some λ.

Processes can encode states. For example, state-like (and separable) process matrix representing Alice and Bob receiving part of an initial state ρ to their labs: $W = \rho^{A_1 B_1} \otimes \mathbb{I}^{A_2 B_2}$.[4]

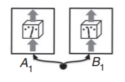
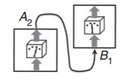
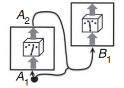
There exist **causally nonseparable** process matrices as well, with $W \neq \lambda W_{A \not\preceq B} + (1-\lambda) W_{B \not\preceq A}$.[5]

[4] Brukner, Costa, and Oreshkov, "Quantum correlations with no causal order".

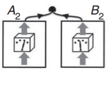
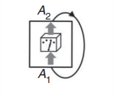
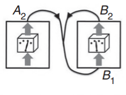
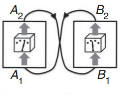
[5] Brukner, "Quantum causality".

Process Matrix Terms

W contains only specific terms, all of which are either $A \not\preceq B$ or $B \not\preceq A$.[6]

$B \not\preceq A$	A_1, B_1, A_1B_1	A_2B_1	$A_1A_2B_1$
$A \not\preceq B$		A_1B_2	$A_1B_1B_2$
Causal order	States	Channels	Channels with memory
			

Forbidden terms lead to non-unit total probability for certain CPTP maps, and correspond to exotic spacetime geometries.[7]

A_2, B_2, A_2B_2	A_1A_2, B_1B_2	$A_1A_2B_2, A_2B_1B_2$	$A_1A_2B_1B_2$
Postselection	Local loops	Channels with local loops	Global loops
			

[6] Brukner, Costa, and Oreshkov, "Quantum correlations with no causal order".
[7] Brukner, Costa, and Oreshkov, "Quantum correlations with no causal order".

Examples

One nonseparable process is $W_1 = \frac{1}{4}\left(\mathbb{I}^{A_1 A_2 B_1 B_2} + \frac{1}{\sqrt{2}}(\sigma_z^{A_2}\sigma_z^{B_1} + \sigma_z^{A_1}\sigma_x^{B_1}\sigma_z^{B_2})\right)$ representing $A \to B$ signalling and $B \to A$ signalling with X-memory, which can violate a causal form of a CHSH inequality, shown in the game below.[8]

Suppose Alice and Bob each have qubits, and flip a coin c when the game begins: on Heads, Bob must signal his qubit state to Alice; on Tails, Alice must signal hers to Bob. The probability of success is
$P_s = P(x = b, c = H) + P(y = a, c = T) = \frac{1}{2}\left(P(x = b|c = H) + P(y = a|c = T)\right)$.

For example: if $A \to B$ then always win whenever $c = T$, but only win half the time when $c = H$. Generally $P_s \leq \frac{3}{4}$ for all separable processes between Alice and Bob.

On the other hand, with process W_1, then $P_s = \frac{2+\sqrt{2}}{4} > \frac{3}{4}$. So causally nonseparable processes can violate measurement inequalities.[9]

[8] Mateus Araújo et al. “Witnessing causal nonseparability”. In: *New J. Phys.* 17 (2015), p. 102001.

[9] Brukner, Costa, and Oreshkov, “Quantum correlations with no causal order”.

Causal Witnesses

Causal witnesses, like entanglement witnesses, prove that a given process is nonseparable. A witness S may be defined to guarantee that $\forall W_{sep}, \mathrm{Tr}(SW_{sep}) \geq 0$. Then, iff $\mathrm{Tr}(SW) < 0$, then W is nonseparable.[10]

The (nonunique) witness depends on the process, found by solving the semidefinite programming problem $\min_S \mathrm{Tr}(SW)$, or equivalently the dual $\min_{\Omega \in \mathcal{W}} \mathrm{Tr}(\Omega)$ (s.t. $W + \Omega$ is separable), then $\mathrm{Tr}(\Omega_{min}) = -d\,\mathrm{Tr}(S_{min}W)$.

Importantly, causal witnesses detect nonseparability even in processes that *don't violate causal inequalities*.[11] Every nonseparable process has at least one witness.

[10] Araújo et al., "Witnessing causal nonseparability".

[11] Araújo et al., "Witnessing causal nonseparability".

Experiment

Range across ρ^{in}, M^A, M^B, use causal witness to show nonseparability.[12]

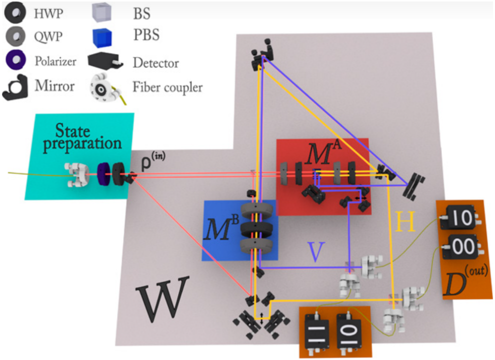


Figure: Photon ρ^{in} passes through beam splitter, then each beam subject to measurement M^A (wave plates and polarizing beam splitter) and unitary M^B (wave plates), but in opposite orders before measuring polarization and position D^{out}.

[12] Giulia Rubino et al. "Experimental verification of an indefinite causal order". In: *Sci. Adv.* 3 (2017), e1602589.

Quantum Switches

Beam splitter example of **quantum switch**, control qubit whose state determines order of operations on other subsystem through unitary $\hat{U}_{CT} = |1\rangle_C \langle 1| \otimes (\hat{A}\hat{B})_T + |0\rangle_C \langle 0| \otimes (\hat{B}\hat{A})_T$.[a]

By preparing quantum switch initialization $|+\rangle_C$, then process on $|\psi\rangle_T$ is superposition of $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$.

[a]Giulio Chiribella et al. "Quantum computations without definite causal structure". In: *Phys. Rev. A* 88 (2 2013), p. 022318.

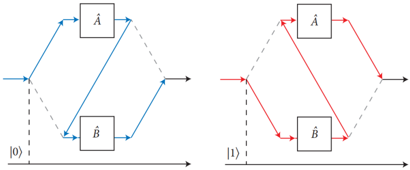


Figure: Brukner, "Quantum causality"

While quantum switches can reduce number of queries necessary for computation, they can't violate causal inequalities.[a] Fundamental difference between causal nonseperability and entanglement.

[a]Araújo et al., "Witnessing causal nonseparability".

Computation Uses

If one knows that A and B either commute or anticommute, then preparing $|+\rangle_C \otimes |\psi\rangle_T$ on a quantum switch can determine commutation with only 1 query, by finally measuring σ_x^C. Separably ordered quantum computation would need at least 2 queries.[13]

From this advantage, quantum switches allow an exponential speedup in Exchange Evaluation games to communicate bit-strings.[14]

Indefinite causal order can also *perfectly* transmit quantum information through a noisy channel with zero-capacity, by communicating in a superposition of time-orders. This contrasts with superpositions of paths through zero-capacity channels.[15]

[13] Chiribella et al., "Quantum computations without definite causal structure".

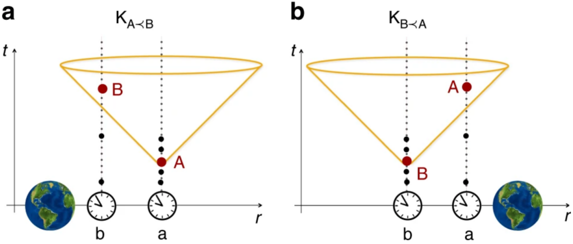
[14] Kejin Wei et al. "Experimental Quantum Switching for Exponentially Superior Quantum Communication Complexity". In: *Phys. Rev. Lett.* 122 (12 2019), p. 120504.

[15] Giulio Chiribella et al. "Indefinite causal order enables perfect quantum communication with zero capacity channels". In: *New J. Phys.* 23 (2021), p. 033039.

Quantum Causality and General Relativity

Gravitational time-dilation implies that proximity to masses slows the local passage of time. So a long-distance spatial superposition between Alice and Bob's labs, with one nearby to the Earth and the other far away, can produce indefinite causal order.

The Earth's gravitational field becomes a quantum switch, superpositionally placing Alice in the light-cone of Bob or vice versa. A photon transmitted between Alice and Bob will experience indefinitely ordered M^A, M^B as in previous experiment.[16]



[16] Magdalena Zych et al. "Bell's theorem for temporal order". In: *Nat. Commun.* 10 (2019), p. 3772.