Lecture 1: Mathematical Preliminaries and Optimality Conditions for Unconstrained Optimization

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Mathematical Preliminaries

In this course, we will focus on the real n-dimensional vector space $\mathbb{R}^n$ and the space of real valued $m \times n$ matrices $\mathbb{R}^{m \times n}$.

Important Subsets of $\mathbb{R}^n$

- $\rightarrow$ Nonnegative orthant: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \ldots, n\}$.
- $\rightarrow$ Positive orthant: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, \ldots, n\}$.
- $\rightarrow$ The closed line segment between $x, y \in \mathbb{R}^n$:

$$[x, y] = \{x + \alpha(y - x) : \alpha \in [0, 1]\}.$$

- $\rightarrow$ The open line segment between $x, y \in \mathbb{R}^n$:

$$[x, y] = \{x + \alpha(y - x) : \alpha \in (0, 1)\}$$

for $x \neq y$ and $(x, x) = \emptyset$.

- $\rightarrow$ Unit simplex: $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, e^T x = 1\}$.

Inner Products

An inner product on $\mathbb{R}^n$ is a map $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:

1. (symmetry) $\langle x , y \rangle = \langle y , x \rangle, \ \forall x, y \in \mathbb{R}^n$.
2. (additivity) $\langle x , y + z \rangle = \langle x , y \rangle + \langle x , z \rangle, \ \forall x, y, z \in \mathbb{R}^n$.
3. (homogeneity) $\langle \lambda x , y \rangle = \lambda \langle x , y \rangle, \ \forall \lambda \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$.
4. (positive definiteness) $\langle x , x \rangle \geq 0, \ \forall x \in \mathbb{R}^n$ and $\langle x , x \rangle = 0$ if and only if $x = 0$.

Example: the "dot product"

$$\langle x , y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i, \ \forall x, y \in \mathbb{R}^n.$$

Vector Norms

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ satisfying:

1. (Nonnegativity) $\|x\| \geq 0, \ \forall x \in \mathbb{R}^n$ and $\|x\| = 0$ if and only if $x = 0$.
2. (positive homogeneity) $\|\lambda x\| = |\lambda| \|x\|, \ \forall x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
3. (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in \mathbb{R}^n$.

→ One natural way to generate a norm on $\mathbb{R}^n$ is to take any inner product $\langle \cdot , \cdot \rangle$ defined on $\mathbb{R}^n$, and define the associated norm

$$\|x\| = \sqrt{\langle x , x \rangle}, \ \forall x \in \mathbb{R}^n.$$

→ For example, the Euclidean norm or l_2-norm:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \ \forall x \in \mathbb{R}^n.$$

l_p-norms

→ The l_p-norm ($p \geq 1$) is defined by $\|x\|_p \equiv \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$.

→ The l_∞-norm is

$$\|x\|_\infty \equiv \max_{i=1,2,\ldots,n} |x_i|.$$

→ It can be shown that

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

The Cauchy-Schwartz Inequality

For all $x, y \in \mathbb{R}^n$,

$$\left| x^T y \right| \leq \|x\| \cdot \|y\|.$$

Matrix Norms

Definition. A norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ is a function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}_+$ satisfying

1. (nonnegativity) $\|A\| \geq 0$ for any $A \in \mathbb{R}^{m\times n}$ and $\|A\| = 0$ if and only if $A = 0$.
2. (positive homogeneity) $\|\lambda A\| = |\lambda|\|A\|$ for any $A \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$.
3. (triangle inequality) $\|A + B\| \leq \|A\| + \|B\|$ for any $A, B \in \mathbb{R}^{m\times n}$.

Induced Norms

→ Given a matrix $A \in \mathbb{R}^{m \times n}$ and two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the induced matrix norm $\|A\|_{a,b}$ (called (a, b)-norm) is defined by

$$\|A\|_{a,b} = \max_x \{\|Ax\|_b : \|x\|_a \leq 1\}.$$

→ By definition, we have

$$\|Ax\|_b \leq \|A\|_{a,b}\|x\|_a.$$

→ An induced norm is a norm.

Matrix Norms Contd

→ *spectral norm*: If $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_2$, the induced $(2,2)$-norm of a matrix $A \in \mathbb{R}^{m\times n}$ is the maximum singular value of A:

$$\|A\|_2 = \|A\|_{2,2} = \sqrt{\lambda_{\max}(A^T A)} \equiv \sigma_{\max}(A).$$

→ 1-*norm*: when $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_1$, the induced $(1,1)$-norm of a matrix $A \in \mathbb{R}^{m\times n}$ is given by (*maximum absolute column sum*)

$$\|A\|_1 = \max_{j=1,2,\ldots,n} \sum_{i=1}^{m} \left|A_{i,j}\right|.$$

→ ∞-*norm*: when $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_\infty$, the induced (∞,∞)-norm of a matrix $A \in \mathbb{R}^{m\times n}$ is given by (*maximum absolute row sum*)

$$\|A\|_\infty = \max_{i=1,2,\ldots,n} \sum_{j=1}^{m} \left|A_{i,j}\right|.$$

The Frobenius Norm

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}, \; A \in \mathbb{R}^{m\times n},$$

which is not an induced norm (why?).

Basic Topological Concepts

→ The open ball with center $c \in \mathbb{R}^n$ and radius r:

$$B(c, r) = \{x : \|x - c\| < r\}.$$

→ The closed ball with center $c \in \mathbb{R}^n$ and radius r:

$$B[c, r] = \{x : \|x - c\| \leq r\}.$$

Definition. Given a set $U \subseteq \mathbb{R}^n$, a point $c \in U$ is called an interior point of U if there exists $r > 0$ for which $B(c, r) \subseteq U$.

→ The set of all interior points of a given set U is called the interior of the set and is denoted by $\text{int}(U)$:

$$\text{int}(U) = \{x \in U : B(x, r) \subseteq U \text{ for some } r > 0\}.$$

→ Example: $\text{int}(B[c, r]) = B(c, r)$

Open and Closed Sets

- $\rightarrow$ An open set is a set that contains only interior points, meaning that $U = \text{int}(U)$. For example, open balls and the positive orthant $\mathbb{R}^n_{++}$.
- $\rightarrow$ A union of any number of open sets is an open set and the intersection of a finite number of open sets is open.
- $\rightarrow$ A set $U \subseteq \mathbb{R}^n$ is closed if it contains all the limits of convergent sequences of vectors in U, i.e., if $\{x_i\}_{i=1}^{\infty} \subseteq U$ satisfies $x_i \rightarrow x^*$ as $i \rightarrow \infty$, then $x^* \in U$.
- $\rightarrow$ U is closed iff its complement U^c is open.
- $\rightarrow$ Examples of closed sets: the closed ball $B[c, r]$, closed line segments, the nonnegative orthant $\mathbb{R}^n_+$ and the unit simplex $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, e^T x = 1\}$.

Boundary Points

Definition. Given a set $U \subseteq \mathbb{R}^n$, a boundary point of U is a vector $x \in \mathbb{R}^n$ satisfying the following: any neighborhood of x contains at least one point in U and at least one point in its complement U^c.

→ The set of all boundary points of a set U is denoted by $\text{bd}(U)$.

→ Examples:

$$\text{bd}(B(c,r)) = \text{bd}(B[c,r]) = \{x \in \mathbb{R}^n : \|x - c\| = r\}$$

$$\text{bd}(\mathbb{R}^n_{++}) = \text{bd}(\mathbb{R}^n_{+}) = ?$$

$$\text{bd}(\mathbb{R}^n) = ?$$

$$\text{bd}(\Delta_n) = ?$$

Closure

→ The closure of a set $U \subseteq \mathbb{R}^n$ is denoted by $\mathrm{cl}(U)$ and is defined to be the smallest closed set containing U:

$$\mathrm{cl}(U) = \bigcap \{T : U \subseteq T,\ T \text{ is closed}\}.$$

→ Equivalently,

$$\mathrm{cl}(U) = U \cup \mathrm{bd}(U).$$

Boundedness and Compactness

- → A set $U \subseteq \mathbb{R}^n$ is called bounded if there exists $M > 0$ such that $U \subseteq B(0, M)$.
- → A set $U \subseteq \mathbb{R}^n$ is called compact if it is closed and bounded.
- → Examples of compact sets: closed balls, unit simplex, closed line segments.

Directional Derivatives and Gradients

Definition. Let f be a function defined on a set $S \subseteq \mathbb{R}^n$. Let $x \in \text{int}(S)$ and let $d \in \mathbb{R}^n$. If the limit

$$\lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}$$

exists, then it is called the directional derivative of f at x along the direction d and is denoted by $f'(x; d)$.

Directional Derivatives and Gradients

→ For any $i = 1, 2, \ldots, n$, if the limit

$$\lim_{t \to 0^+} \frac{f(x + te_i) - f(x)}{t}$$

exists, then its value is called the i-th partial derivative and is denoted by $\frac{\partial f}{\partial x_i}(x)$.

→ If all the partial derivatives of a function f exist at a point $x \in \mathbb{R}^n$, then the gradient of f at x is

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

Continuous Differentiability

A function f defined on an open set $U \subseteq \mathbb{R}^n$ is called continuously differentiable over U if all the partial derivatives exist and are continuous on U. In that case,

$$f'(x; d) = \nabla f(x)^T d, \ \forall x \in U, \ d \in \mathbb{R}^n.$$

Proposition. Let $f : U \to \mathbb{R}$ be defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is continuously differentiable over U. Then

$$\lim_{d \to 0} \frac{f(x+d) - f(x) - \nabla f(x)^T d}{\|d\|} = 0, \ \forall x \in U.$$

Equivalently, we can write the above result as follows:

$$f(y) = f(x) + \nabla f(x)^T (y - x) + o(\|y - x\|),$$

where $o(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}$ is a 1-D function satisfying $\frac{o(t)}{t} \to 0$ as $t \to 0^+$.

Twice Differentiability

→ The partial derivatives $\frac{\partial f}{\partial x_i}$ are themselves real-valued functions that can be partially differentiated. The (i, j)-partial derivatives of f at $x \in U$ (if exists) is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial \left(\frac{\partial f}{\partial x_j}\right)}{\partial x_i}(x).$$

→ A function f defined on an open set $U \subseteq \mathbb{R}^n$ is called twice continuously differentiable over U if all the second-order partial derivatives exist and are continuous over U. In that case, for any $i \neq j$ and any $x \in U$:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x).$$

The Hessian

The Hessian of f at a point $x \in U$ is the $n \times n$ matrix:

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

For twice continuously differentiable functions, the Hessian is a symmetric matrix.

Linear Approximation Theorem

Theorem. Let $f : U \to \mathbb{R}$ be defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U. Let $x \in U$ and $r > 0$ satisfy $B(x, r) \subseteq U$. Then $\forall y \in B(x, r)$ there exists $\xi \in [x, y]$ such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(\xi)(y - x).$$

Quadratic Approximation Theorem

Theorem. Let $f : U \to \mathbb{R}$ be defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U. Let $x \in U$ and $r > 0$ satisfy $B(x, r) \subseteq U$. Then $\forall y \in B(x, r)$,

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x) + o(\|y - x\|^2).$$

Optimality Conditions for Unconstrained Optimization

Global Minima

Definition. Let $f : S \to \mathbb{R}$ be defined on a set $S \subseteq \mathbb{R}^n$. Then

1. $x^* \in S$ is a global minimum point of f over S if $f(x^*) \leq f(x),\ \forall x \in S$.
2. $x^* \in S$ is a strict global minimum point of f over S if $f(x^*) < f(x),\ \forall x^* \neq x \in S$.

Definition. The minimum value of f over S is defined as

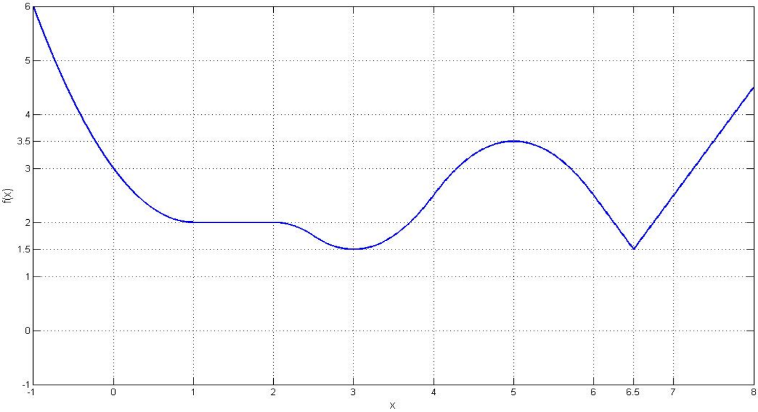
$$\inf\{f(x) : x \in S\}.$$

Local Minima

Definition. Let $f : S \to \mathbb{R}$ be defined on a set $S \subseteq \mathbb{R}^n$. Then

1. $x^* \in S$ is a local minimum of f over S if there exists $r > 0$ for which $f(x^*) \le f(x), \ \forall x \in S \bigcap B(x^*, r)$.
2. $x^* \in S$ is a strict local minimum of f over S if there exists $r > 0$ for which $f(x^*) < f(x), \ \forall x^* \ne x \in S \bigcap B(x^*, r)$.

Example: classify all the global and local optima



Fermat's Theorem - First-Order Optimality Condition

Theorem. Let $f : U \to \mathbb{R}$ be a function defined on a set $U \subset \mathbb{R}^n$. Suppose that $x^* \in \text{int}(U)$ is a local optimum point and that all the partial derivatives of f exist at x^*. Then $\nabla f(x^*) = 0$.

Proof. Consider the 1-D function $g(t) = f(x^* + te_i)$. □

Stationary Points

Definition. Let $f : U \longrightarrow \mathbb{R}$ be a function defined on a set $U \subset \mathbb{R}^n$. Suppose that $x^* \in \mathrm{int}(U)$ and that all the partial derivatives of f are defined at x^*. Then x^* is called a stationary point of f if $\nabla f(x^*) = 0$.

Classification of Matrices - Positive Definiteness

- → A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive semidefinite, denoted by $A \succeq 0$, if $x^T A x \geq 0, \ \forall x \in \mathbb{R}^n$.
- → A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive definite, denoted by $A \succ 0$, if $x^T A x > 0, \ \forall 0 \neq x \in \mathbb{R}^n$.
- → A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called indefinite, if there exists $x, y \in \mathbb{R}^n$ such that $x^T A x > 0, \ y^T A y < 0$.

The Principal Minors Criteria

Definition. Given an $n \times n$ matrix, the determinant of the upper left $k \times k$ submatrix is called the k-th principal minor and is denoted by $D_k(A)$. For example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$$D_1(A) = a_{11},\ D_2(A) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ D_3(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Theorem (principal minors criteria). Let A be an $n \times n$ symmetric matrix. Then A is positive definite if and only if $D_1(A) > 0, D_2(A) > 0, \ldots, D_n(A) > 0.$

Diagonal Dominance

Definition. Let A be a symmetric $n \times n$ matrix.

(a). A is called diagonally dominant if

$$|A_{ii}| \geq \sum_{j \neq i} \left|A_{ij}\right|, \ \forall i = 1, 2, \ldots, n$$

(b). A is called strictly diagonally dominant if

$$|A_{ii}| > \sum_{j \neq i} \left|A_{ij}\right|, \ \forall i = 1, 2, \ldots, n$$

Theorem (positive (semi)definiteness of diagonally dominant matrices).

(a). If A is symmetric, diagonally dominant with nonnegative diagonal elements, then A is positive semidefinite.

(b). If A is symmetric, **strictly** diagonally dominant with **positive** diagonal elements, then A is positive definite.

Necessary Second-Order Optimality Conditions

Theorem. Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U and that x^* is a stationary point. Then if x^* is a local minimum point, then $\nabla^2 f(x^*) \succeq 0$.

Proof. Use the Quadratic Approximation Theorem. □

Sufficient Second-Order Optimality Conditions

Theorem. Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U and that x^* is a stationary point. Then if $\nabla^2 f(x^*) \succ 0$, x^* is a strict local minimum point of f over U.

Proof. Since Hessian is continuous, there exists a ball $B(x^*, r) \subseteq U$ for which $\nabla^2 f(x) \succ 0, \ \forall x \in B(x^*, r)$. By the Linear Approximation Theorem, there exists a vector $z_x \in [x^*, x]$ (and hence $z_x \in B(x^*, r)$) for which

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)^T \nabla^2 f(z_x)(x - x^*).$$

$\nabla^2 f(z_x) \succ 0 \Rightarrow$ for any $x \in B(x^*, r)$ such that $x \neq x^*$, the inequality $f(x) > f(x^*)$ holds, implying that x^* is a strict local minimum point of f over U. □

Saddle Points

Definition. Let $f : U \to \mathbb{R}$ be a continuously differentiable function defined on an open set $U \subseteq \mathbb{R}^n$. A stationary point $x^* \in U$ is called a saddle point of f over U if it is neither a local mimimum point nor a local maximum point of f over U.

Theorem (sufficient condition for saddle points). Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U and that x^* is a stationary point. if $\nabla^2 f(x^*)$ is an indefinite matrix, then x^* is a saddle point of f over U.

Attainment of Minimal / Maximal Points

Theorem (Weierstrass). Let f be a continuous function defined over a nonempty compact set $C \subseteq \mathbb{R}^n$. Then there exists a global minimum point of f over C and a global maximum point of f over C.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function over $\mathbb{R}^n$. f is called coercive if

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

Theorem. Let Let $f : \mathbb{R}^n \to \mathbb{R}$ be a coercive and continuous function and let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then f attains a global minimum point on S.

Proof. → Pick any $x_0 \in S$. f being coercive $\Rightarrow \exists M > 0$ such that

$$f(x) > f(x_0) \text{ for any } x \text{ such that } \|x\| > M.$$

→ Since any global minimizer x^* of f over S satisfies $f(x^*) \leq f(x_0)$, it follows that the set of global minimizer of f over S is the same as the set of global minimizers of f over $S \cap B[0, M]$.

→ The set $S \cap B[0, M]$ is compact and nonempty $\Rightarrow$ (by the Weierstrass theorem) $\exists$ a global minimizer of f over $S \cap B[0, M]$ and hence also over S.

□

Example

Classify the stationary points of the function $f(x_1, x_2) = -2x_1^2 + x_1 x_2^2 + 4x_1^4$.

$$\nabla f(x) = \begin{pmatrix} -4x_1 + x_2^2 + 16x_1^3 \\ 2x_1 x_2 \end{pmatrix}, \quad \nabla^2 f(x_1, x_2) = \begin{pmatrix} -4 + 48x_1^2 & 2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}.$$

⇒ stationary points are solutions to

$$\begin{aligned} -4x_1 + x_2^2 + 16x_1^3 &= 0, \\ 2x_1 x_2 &= 0. \end{aligned}$$

⇒ stationary points are $(0, 0)$, $(0.5, 0)$, $(-0.5, 0)$.

$$\nabla^2 f(0.5, 0) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nabla^2 f(-0.5, 0) = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \nabla^2 f(0, 0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$$

strict local minimum   saddle point   saddle point (why?)

Global Optimality Conditions

Theorem. Let f be a twice continuously defined over $\mathbb{R}^n$. Suppose that $\nabla^2 f(x) \succeq 0,\ \forall x \in \mathbb{R}^n$. Let $x^* \in \mathbb{R}^n$ be a stationary point of f. Then x^* is a global minimum point of f.

Proof. By the Linear Approximation Theorem, it follows that for any $x \in \mathbb{R}^n$, there exists a vector $z_x \in [x^*, x]$ for which

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)^T \nabla^2 f(z_x)(x - x^*).$$

Since $\nabla^2 f(z_x) \succeq 0$, we have that $f(x) \geq f(x^*)$, which implies that x^* is a global minimum point of f. $\square$

Quadratic Functions

A quadratic function over $\mathbb{R}^n$ is a function of the form

$$f(x) = x^T A x + 2b^T x + c,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Its gradient and Hessian can be easily obtained (exercise):

$$\nabla f(x) = 2Ax + 2b,$$
$$\nabla^2 f(x) = 2A.$$

Lemma. Let $f(x)$ be a quadratic function. Then

1. x is a stationary point of f iff $Ax = -b$.
2. If $A \succeq 0$, then x is a global minimum point of f iff $Ax = -b$.
3. If $A \succ 0$, then $x = -A^{-1}b$ is a strict global minimum point of f.

Coerciveness of Quadratic Functions

Lemma. Let $f(x) = x^T Ax + 2b^T x + c$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then f is coercive if and only if $A \succ 0$.

Characterization of the Nonnegativity of Quadratic Functions

Theorem. Let $f(x) = x^T Ax + 2b^T x + c$, where $A \in \mathbb{R}^{n\times n}$ is symmetric, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following two claims are equivalent:

(a). $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

(b). $\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \succeq 0$.

Exercises

Beck: 1.2, 1.14, 2.1, 2.2, 2.9, 2.14, 2.17.v