Extra Slides

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Quadratic Function

To find the Hessian matrix of the given quadratic function $f(x_1, x_2) = 1000x_1^2 + 40x_1x_2 + x_2^2$, we need to compute the second-order partial derivatives with respect to x_1 and x_2.

Hessian Matrix Definition

The Hessian matrix H of a function f is a square matrix of second-order partial derivatives, defined as follows:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \tag{1}$$

First-Order Partial Derivatives

First, let's compute the first-order partial derivatives:

$$\frac{\partial f}{\partial x_1} = 2000x_1 + 40x_2 \tag{2}$$

$$\frac{\partial f}{\partial x_2} = 40x_1 + 2x_2 \tag{3}$$

Second-Order Partial Derivatives

Next, we compute the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}(2000x_1 + 40x_2) = 2000 \tag{4}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}(2000x_1 + 40x_2) = 40 \tag{5}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_1}(40x_1 + 2x_2) = 40 \tag{6}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(40x_1 + 2x_2) = 2 \tag{7}$$

Hessian Matrix

Thus, the Hessian matrix H is:

$$H = \begin{bmatrix} 2000 & 40 \\ 40 & 2 \end{bmatrix} \tag{8}$$

Positive Definite Matrix

A positive definite matrix is a specific type of matrix that has important properties in various fields such as mathematics, statistics, optimization, and physics. Here are the key characteristics and definitions:
A positive definite matrix A is always symmetric, meaning

$$A = A^T.$$

A symmetric matrix A is positive definite if for any non-zero vector x, the quadratic form $x^T Ax$ is strictly positive. Mathematically, this can be expressed as:

$$x^T Ax > 0 \quad \text{for all} \quad x \neq 0.$$

All the eigenvalues of a positive definite matrix are positive.