UNM Physics 405: Lecture 7: Gauss' Law

In principle, we are done with electrostatics

\[ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_0^R \vec{r}' \cdot \rho(\vec{r}') \frac{d^3r'}{R^2} \]

These integrals are generally very nasty (e.g. sphere of charge). We need a bag of tricks.

Integral form of Gauss' Law

Consider the flux of the electric field associated with a point charge through a "imaginary sphere" with center \( Q \).

\[ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \]

\[ \int \vec{E} \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \int d\sigma = \frac{Q}{4\pi\epsilon_0 R^2} \]

The integral is independent of \( R \).
In fact, the flux of $\vec{E}$ is independent of the shape of $S$.

\[
\hat{E} \cdot d\hat{a} = \frac{q}{4\pi \varepsilon_0 \frac{r_0^2}{r^2}} (\hat{r} \cdot \hat{n})
\]

\[
d\hat{a} = \hat{n} \, da
\]

\[
da' = da \cos \theta = r_0^2 d\Omega \cos \theta
\]

\[
\Rightarrow da = \frac{r_0^2 d\Omega}{\cos \theta}
\]

\[
\Rightarrow \int \hat{E} \cdot d\hat{a} = \frac{d\Omega}{4\pi} \frac{q}{\varepsilon_0} \quad \text{if } q \text{ is inside surface}
\]

If $q$ is outside surface,

Since flux independent of shape,

\[
\text{flux in} = \text{flux out}
\]

\[
\Rightarrow \oint \hat{E} \cdot d\hat{a} = 0
\]
Superposition Principle

Given $N$-charges inside a closed surface

$$\oint \vec{E} \cdot d\vec{a} = \int (\vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N) \cdot d\vec{a} = \sum_{i=1}^{N} \frac{q_i}{\epsilon_0} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Gauss's Law: The flux of $\vec{E}$ through any closed surface $= \frac{Q_{\text{enclosed}}}{\epsilon_0}$ (the total charge enclosed inside)

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

- Gauss's law is a consequence of the $\frac{1}{r^2}$ nature of the force $\Rightarrow$ geometrical.
The "number of field lines piercing out of surface" = # of sources inside the volume.

- Gauss's law is always true, but it is not always useful as a tool for calculating the $E$-field.

Use Gauss's law for charge distributions which have a high degree of symmetry.
Examples

1. Sphere of charge of radius \( R \) which has a charge \( Q \) uniformly distributed throughout.

   Symmetry: Spherical \( \Rightarrow \vec{E}(\vec{r}) = E(\vec{r})\hat{\vec{r}} \)
   
   - \( E \) points in radial direction
   - \( |E| \) depends only on \( r \) (not \( \theta \) and \( \phi \))

   Choose a "Gauss's surface" so that \( \vec{E} \cdot \vec{d}a = 0 \) is constant of \( S \). Here \( S \) is a sphere.

   \[ r > R \]

   \[ \oint \vec{E} \cdot \vec{d}a = E(r) \oint \vec{d}a = E(r) 4\pi r^2 \]

   \[ = E(r) 4\pi r^2 = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \]

   \[ \Rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{\vec{r}} \quad \text{(field of point charge \( Q \) at \( \text{origin} \))} \]

   \[ r > R \]

   \[ r < R \]

   \[ E(r) 4\pi r^2 = \frac{Q_{\text{enc}}(r)}{\varepsilon_0} \]

   \[ Q_{\text{enc}}(r) = \left( \frac{4}{3} \pi r^3 \right) \rho = \frac{4}{3} \pi r^3 \frac{Q}{R^3} = \frac{Q \rho_0}{R^3} \]

   \[ \Rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{\frac{R^3}{r^3}} \hat{\vec{r}} \]

   \[ r < R \]
(1) Sphere of charge continued

\[ E(r) \]

\[ \left( \frac{1}{4\pi \varepsilon_0} \right) \frac{Q}{r^2} \sim r \sim \frac{1}{r^2} \]

\[ R_0 \]

(2) Spherical shell of charge

\[ \sigma = \frac{Q}{4\pi R^2} \]

The field inside the sphere is zero.

*Note:* This would not be true without symmetry. We used the fact that

\[ \oint E \cdot d\vec{a} = E_{\text{surface}} \oint \sigma d\vec{a} \]

\[ \text{We can have } \oint E \cdot d\vec{a} = 0 \text{ w/o } \]

\[ \vec{E} = 0 \text{ in non-symmetric case.} \]
Example 3: Infinite line charge

\[ \lambda = \text{charge length} \]

Here we know \( \vec{E} = E(r) \hat{r} \) (cylindrical symmetry)

Choose Gaussian surface as cylinder shown above

\[ \oint \vec{E} \cdot d\vec{a} = \int E(r) \, da = E(r) \cdot 2\pi r L \]

surface excluding "caps"

\[ = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{2L}{\varepsilon_0} \]

\[ \Rightarrow E(r) = \frac{2}{2\pi \varepsilon_0 r} \]

falls off as \( \frac{1}{r^2} \)

Example 4: Infinite plane of charge

\[ \sigma = \text{charge area} \]

\[ \Rightarrow \vec{E} = \begin{cases} E_0 \hat{z} & \text{above} \\ -E_0 \hat{z} & \text{below} \end{cases} \]

\[ \Rightarrow \oint \vec{E} \cdot d\vec{A} = E_0 (A_{\text{top}} + A_{\text{bottom}}) = 2A E_0 \]

\[ = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \]
\[ \mathcal{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0} & \text{above} \\ -\frac{\sigma}{2\varepsilon_0} & \text{below} \end{cases} \]

-plane of charge

**Gauss's Law: Differential Form**

Use the divergence theorem:

\[ \oint_{S} \mathbf{E} \cdot d\mathbf{a} = \int_{V} \nabla \cdot \mathbf{E} \, d^3r = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{V} \rho(r) \, d^3r \]

Volume arbitrary \[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

**Meaning:** The local divergence of \( \mathbf{E} \) is given by the local source of electric field \( \mathbf{E} \) \( \nabla \cdot \mathbf{E} \) \( \rho(r) \)

The integral theorem \( \oint_{S} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \)

is a global statement.