Capacitance

An important concept in electrostatics is the capacity for a conductor to hold charge, or the "capacitance".

Consider two conductors held at some potential difference by an external source (battery).

\[ V \]

\[ +Q \quad -Q \]

A small surface charge +Q will develop on one conductor and -Q on the other.

The surface charge is proportional to V.

\[ \Rightarrow \text{Define } C = \frac{Q}{V} = \text{Capacitance} \]

C is purely geometrical.

To calculate C, we place charge +Q, -Q on two conductors and determine the potential difference between them.

Unit of capacitance \([C] = \text{Farad} = \frac{\text{Coul}}{\text{Volt}}\)

1 Farad is huge.
Canonical example: The parallel plate capacitor

We will approximate the plates as infinite in extent, valid when \( d \ll \text{width} \) and \( d \ll \text{length} \)

**Edge on** view:

\[ \begin{align*}
z = 0 & \quad \Rightarrow \quad +Q \text{ on surface} \\
2z = d & \quad \Rightarrow \quad -Q \text{ on surface}
\end{align*} \]

Ignoring "fringing field" \( \vec{E} = \frac{Q}{\varepsilon_0 A} \) (from Gauss' Law)

Potential difference \( V = -\int E \cdot dz = Ed \)

\[ V = \frac{Qd}{\varepsilon_0 A} \quad \Rightarrow \quad C = \frac{Q}{V} \]

\[ \Rightarrow C = \frac{\varepsilon_0 A}{d} \quad \text{Parallel Plate Capacitor} \]

General forms for two conductor geometry:

\[ C = \varepsilon_0 \frac{A}{d} \quad \text{Surface area between distance between conductors} \]

Units: \( \varepsilon_0 = 8.85 \times 10^{-12} \quad \left( \frac{\text{Coul}^2}{\text{m}^2} = \frac{\text{Coul}}{\text{m}} \cdot \frac{\text{m}}{\text{Coul}} = \frac{\text{F}}{\text{m}} \right) \)

\[ \Rightarrow \varepsilon_0 = 8.85 \times 10^{-12} \text{F/m} \quad \text{Pico Farad} \]
Example:

Capacitance of concentric spheres

\[ F = \begin{cases} \frac{Q}{4\pi \varepsilon_0 r^2} & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases} \]

\[ V_+ = -\int_0^b E_+ \cdot dl = -\int_a^b E(r) dr = \frac{Q}{4\pi \varepsilon_0} \int_a^b \frac{dr}{r^2} \]

\[ = \frac{Q}{4\pi \varepsilon_0} \left(-\frac{1}{r}\right)^b_a = \frac{Q}{4\pi \varepsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right) \]

\[ \Rightarrow V_+ = \frac{Q}{4\pi \varepsilon_0} \left(\frac{b - a}{ab}\right) \text{ Exact result} \]

Consider the limit as the spacing between the spheres becomes small. We expect this to approach the geometry of a parallel plate (flat earth society).

**Exact**

\[ C = \frac{Q}{V_+} = 4\pi \varepsilon_0 \frac{ab}{b - a} \]  

**Limit**

\[ C = 4\pi \varepsilon_0 \frac{a(a+d)}{(a+d)-a} = 4\pi \varepsilon_0 \frac{a^2+ad}{d} \]

\[ \lim_{d \to 0} \frac{\varepsilon_0 (4\pi d^2)}{d} \frac{1}{d} = \varepsilon_0 \frac{4}{d} \]
Energy stored in capacitor:

An external source (battery) must do work to move charge from one conductor onto the other (the energy stored in the charge distribution).

\[ dW = dq \left( \frac{q}{C} \right) \Rightarrow \text{Total Work } W = \int dW \]

\[ W = \int_0^Q dq \left( \frac{q}{C} \right) = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV = \frac{1}{2} CV^2 \]

Equivalent to think about energy stored in field

\[ W = U = \frac{\varepsilon_0}{2} \int E^2 \, d^3r \]

Example: Parallel Plate Capacitor

\[ E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A \varepsilon_0} \text{ Uniform inside, zero out} \]

\[ U = \frac{\varepsilon_0}{2} E^2 (Ad) = \frac{1}{2} \frac{Q^2 (d)}{(A \varepsilon_0)} = \frac{1}{2} \frac{Q^2}{C} \]

\[ \uparrow \text{ Volume where } E \neq 0 \]
Example: concentric spheres

\[ V = \frac{\varepsilon_0}{2} \int_0^b E^2 dr = \varepsilon_0 \int_a^b \frac{Q^2}{16\pi \varepsilon_0^2 r^2} \, dr \]

\[ = \frac{Q^2}{8\pi \varepsilon_0} \int_a^b \frac{dr}{a^2 - r^2} = \frac{Q^2}{8\pi \varepsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right) \]

\[ = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \sqrt{V^2} \]

Charging up capacitors

Let us take a small detour away from electrostatics and consider the flow of charge (current) in a circuit.

"RC circuit"

\[ \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (\text{Kirchoff's law}) \]

\[ \mathbf{V} = \mathbf{V}_R + \mathbf{V}_C \]

\[ V_R = I \frac{dQ}{dt} R \quad (\text{Ohm's law}) \]

\[ V_C = \frac{Q}{C} \]

\[ \frac{d}{dt} \left( RC \frac{dQ}{dt} + \frac{Q}{C} \right) = V \]

\[ \frac{dQ}{dt} + \frac{1}{RC} Q = \frac{V}{R} \]

First order differential equation
Solution: \( Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t) \)

Homogeneous solution (set \( V = 0 \))
\[
\frac{dQ_{\text{hom}}}{dt} + \frac{1}{RC} Q_{\text{hom}} = 0 \quad \Rightarrow \quad Q_{\text{hom}}(t) = Ke^{-t/RC}
\]

"RC time constant"

Particular solution = Steady state \( \frac{dQ_{\text{part}}}{dt} = 0 \)
\[
\Rightarrow \frac{1}{RC} Q_{\text{part}} = \frac{V}{R} \quad \Rightarrow \quad Q_{\text{part}} = CV
\]
\[
\Rightarrow \ Q(t) = Ke^{-t/RC} + CV
\]

Find \( K \) from initial condition \( Q(0) \)
\[
\Rightarrow \ Q(0) = K + CV \quad \Rightarrow \quad K = Q(0) - CV
\]

\[
\Rightarrow \ Q(t) = Q(0) e^{-t/RC} + CV(1 - e^{-t/RC})
\]

Decay of initial state

charge up to final state

\( Q(0) \)
\[
Q(t) \qquad CV \quad \cdots \cdots \quad -t
\]