Physics 405 - Lecture 13

Intro to Laplace's Eqn

The fundamental equations of electrostatics are

$$\left\{ \begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned} \right\}$$

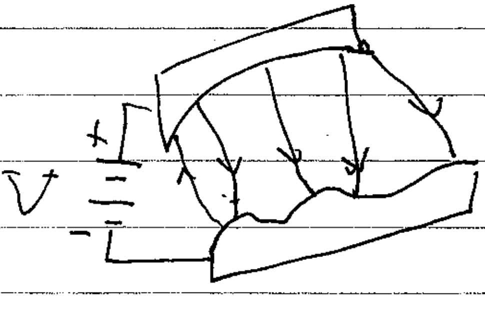
We know $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$

$$\Rightarrow \boxed{\nabla^2 V = -\rho/\epsilon_0} \quad \text{Poissons eqn}$$

With the boundary condition that $V \to 0$ @ infinity, the solution is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

In principle, this is all there is to electrostatics, but in practice one often does not specify the charge density everywhere $\rho(\vec{r})$. Instead, one might specify the potential on some conducting surfaces



The charge density will adjust itself to make this an equipotential, but how?

Problem of electrostatics are thus often of the form of <u>boundary</u> <u>valued</u> <u>problems</u>.

Note that in region where $\rho(\vec{r}) = 0$

$$\boxed{\nabla^2 V = 0} \quad \text{Laplace's Equation}$$

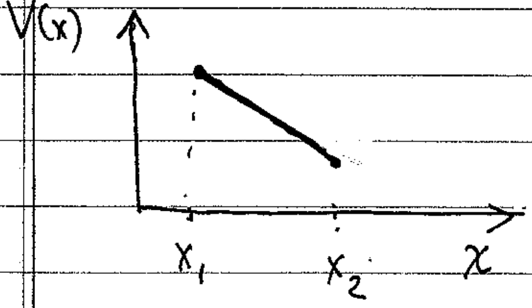
For a large variety of problems of interest, electrostatics is solving Laplaces Eq.
(Also applicable in variety of other physical contexts, e.g., gravitation, heat flow, elastic surfaces, etc.)

Nature of solutions:

Example: 1D $\quad \frac{d^2V}{dx^2} = 0$

2^{nd} order Ordinary diff'eqn. Requires two constants to specify solution (boundary conditions)

$$V(x) = C_1 x + C_2 \quad \text{(General solution)}$$



V(x) is completely specified by its value at two points, e.g. at the boundary $V(x_1)$ and $V(x_2)$.

Alternatively we can specify the slope at one point and value at another.

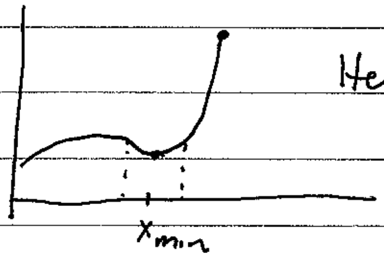
General Properties from 1D

(1) $V(x)$ specified by values at boundary or ~~derivat~~ derivative at boundary (up to overall choice of ground).

(2) $V(x)$ is the average of the values surrounding

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

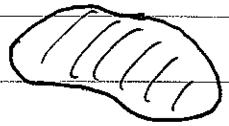
(3) $V(x)$ has no local max or min, except at boundaries



Here $V(x_{mix})$ is not the ~~max or min~~ average of surrounding values

The properties generalize to higher dimensions

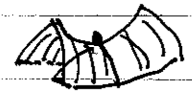
2D: E.g. elastic sheet (ignoring gravity). Shape of sheet determined solely by shape of bounding contour



$V(x,y)$ can have no local max or min:

$$\frac{\partial^3 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \Rightarrow \text{If } \frac{\partial^2 V}{\partial x^2} < 0, \quad \frac{\partial^2 V}{\partial y^2} > 0$$

Saddle

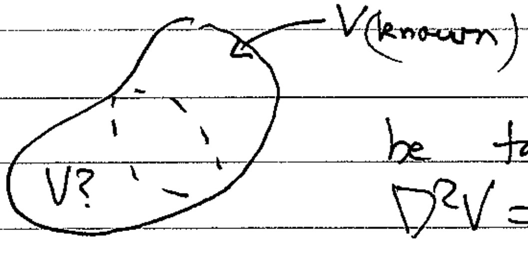


In 3D: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

- Solution completely specified by $V\big|_{\text{boundary}}^{(\text{Dirchelet})}$ or $\hat{n}\cdot\vec{\nabla}V\big|_{\text{boundary}}^{(\text{Neumann})}$ (overspecify to give both everywhere)
- V has no local max or min (saddles only)
- $V(\vec{x})$ is average of values surrounding it (can be use to numerically find solution)

<u>Uniqueness</u> Only one solution fits the boundaries

Proof: Suppose $V(\vec{r})$ is specified everywhere on bounding surfaces (some surfaces can be @ ∞).



Let $V_1(\vec{r})$ and $V_2(\vec{r})$ be two different solutions to $\nabla^2 V = 0$ with same boundary conditions

Let $U \equiv V_1 - V_2 \Rightarrow \nabla^2 U = \nabla^2 V_1 - \nabla^2 V_2 = 0$
and $U = 0$ on boundary

But U cannot have any local max or min

$\Rightarrow U = 0$ everywhere

$\Rightarrow \boxed{V_1(\vec{r}) = V_2(\vec{r})}$ q.e.d.

Further remarks

- The uniqueness theorem extends to solutions of Poisson's Eqn.

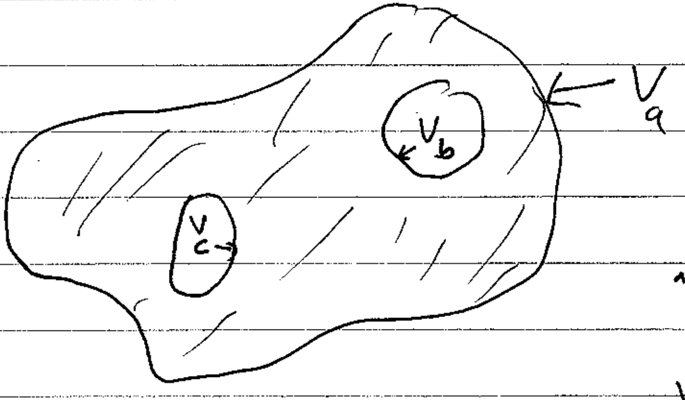
Suppose $V_1(\vec{r})$ and $V_2(\vec{r})$ are two solutions to $\nabla^2 V = -\rho/\epsilon_0$ with same value on bounding surface

$$\text{Let } U = V_1 - V_2 \Rightarrow \nabla^2 U = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

$$\Rightarrow U = 0 \quad \Rightarrow \quad V_1(\vec{r}) = V_2(\vec{r})$$

$\Rightarrow$ Potential is uniquely specified given $\rho(\vec{r})$ through volume of interest and value of $V(\vec{r})$ on the bounding surfaces

- The ~~[illegible]~~ volume need not be "simply connected"

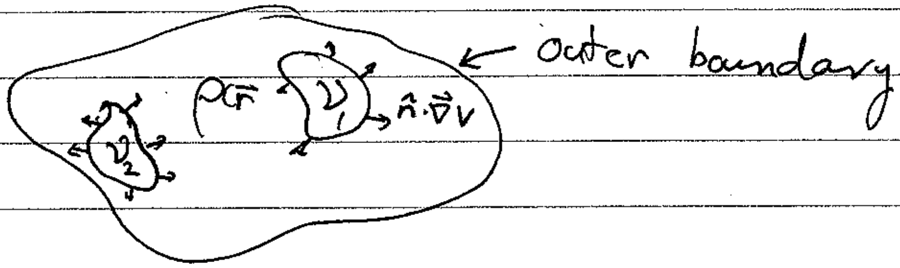


e.g.
Volume with "cavities".

- V_a on outer ~~[illegible]~~ surface
- V_b and V_c on cavity surfaces

Second uniqueness theorem

The potential is uniquely specified by the value of $\hat{n}\cdot\vec{\nabla}V$ on all surfaces and the charge density throughout the volume.



Consider $\oint d\vec{a}\cdot\hat{n}\vec{\nabla}V$ on surfaces

$$= -\oint d\vec{a}\cdot\vec{E} = \frac{-Q_{enc}}{\epsilon_0}$$

Again consider two possible solutions $\vec{E}_1$ and $\vec{E}_2$.

Let $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\Rightarrow \vec{\nabla}\cdot\vec{E}_3 = 0 \quad \text{and} \quad \oint \vec{E}_3\cdot d\vec{a} = 0$$

Now ~~consider~~ consider $\vec{\nabla}\cdot(V_3(\vec{r})\vec{E}_3(\vec{r}))$

$$= V_3(\vec{r})\left(\vec{\nabla}\cdot\vec{E}_3\right) + E_3\cdot(\vec{\nabla}V_3) = -\left(\vec{E}_3(\vec{r})\right)^2$$

$$\Rightarrow \int_{Vol} \vec{\nabla}\cdot(V_3\vec{E}_3) = \oint_{Surface} V_3\vec{E}_3\cdot d\vec{a} = -\int_{Vol}|\vec{E}_3|^2 d^3r$$

Now, $\vec{E}_3 \cdot d\vec{a} = da\,\hat{n} \cdot \vec{\nabla} V_3$

and $\hat{n} \cdot \vec{\nabla} V_3 = \hat{n} \cdot \vec{\nabla} V_1 - \hat{n} \vec{\nabla} V_2$

$= 0$ on specified surface

$$\Rightarrow \int_{Vol} |\vec{E}_3|^2 \, d^3r = 0$$

but $\int |\vec{E}_3|^2 d^3r > 0$ unless $|\vec{E}_3|^2 = 0$ everywhere

$$\Rightarrow \boxed{\vec{E}_1(\vec{r}) = \vec{E}_2(\vec{r})} \quad \text{Q.E.D.}$$

Note: Specifying $\hat{n} \cdot \vec{\nabla} V$ on the surface of a conductor is equivalent to specifying the charge density on the surface since $\vec{E} = 0$ inside the conductor and by the boundary conditions

$$\hat{n} \cdot \vec{E}_{out} - \underbrace{\hat{n} \cdot \vec{E}_{in}}_{0} = \sigma/\epsilon_0$$

$$\Rightarrow \boxed{\hat{n} \cdot \vec{\nabla} V \Big|_{surface} = -\sigma/\epsilon_0}$$